PLAN

- **Introduction**
- **Steady Stokes problem and Uzawa algorithm**
- **Unsteady Stokes problem and Uzawa algorithm**
- **Time splitting and spectral element for the Unsteady Stokes problem**
- **Parallel Domain Decomposition Goda scheme.**
- **Conclusion.**

The time-dependent Stokes problem

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + \text{grad}\, p &= \mathbf{f} \qquad \Omega_t, \\
\text{div}\, \mathbf{u} &= 0 \qquad \Omega_t, \\
\mathbf{u} &= 0 \qquad \partial\Omega_t,
\end{aligned}
$$

- $\mathbf{u}$: velocity field,
- p : pressure field,
- $\mathbf{f}$: body force field,
- Re : Reynolds number,
- Ω : bi- or three-dimensional domain,
- $[0, t^*]$: time interval.
- $\Omega_t = \Omega \times]0, t^*[$

First part : Steady Stokes problem (Re = 1)

The continuous problem is :

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{dans } \Omega, \\ \text{div } \mathbf{u} &= 0 \qquad \text{dans } \Omega, \\ \mathbf{u} &= 0 \qquad \text{sur } \partial\Omega. \end{aligned}$$

The weak problem is : find $(\mathbf{u}, p) \in X \times M$ such that:

$$\begin{aligned} (\text{grad } \mathbf{u}, \text{grad } \mathbf{v}) - (p, \text{div } \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in X, \\ -(q, \text{div } \mathbf{u}) &= 0 \qquad \forall q \in M. \end{aligned}$$

- $X = H_0^1(\Omega)^d$,
- $M = L_0^2(\Omega)$,

$\Longrightarrow$ Well posed problem

Spectral method for Stokes problem

The discret spaces are :

- $X_N = P_N^0(\Omega)^d$,
- $\mathcal{P}_\mathcal{M} = P_M \cap L_0^2(\Omega)$,

Find $\mathbf{u}_N$ in X_N and p_N in $\mathcal{P}_\mathcal{M}$ such that:

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}_N) + b_N(p_N, \mathbf{v}_N) &= (\mathbf{f}, \mathbf{v})_N \ \ \forall \mathbf{v}_N \in X_N, \\ b_N(q_N, \mathbf{u}_N) &= 0 \qquad\qquad \forall q_N \in \mathcal{P}_\mathcal{M}. \end{aligned}$$

where

$$\begin{aligned} a_N(\mathbf{u}_N, \mathbf{v}_N) &= (\text{grad } \mathbf{u}_N, \text{grad } \mathbf{v})_N \\ b_N(q_N, \mathbf{v}_N) &= -(q_N, \text{div } \mathbf{v}_N)_N \end{aligned}$$

The discrete scalar product on $P_N(\Omega)$ is defined by

$$(u, v)_N = \begin{cases} \displaystyle\sum_{i,j=0}^{N} u(\xi_i, \xi_j) v(\xi_i, \xi_j) \rho_i \rho_j, \\ \displaystyle\sum_{i,j,k=0}^{N} u(\xi_i, \xi_j, \xi_k) v(\xi_i, \xi_j, \xi_k) \rho_i \rho_j \rho_k \end{cases}$$

What space for the pressure ?

$\mathcal{P}_{\mathcal{M}}$ must be shosen such that :

$$\inf_{q_N \in \mathcal{P}_{\mathcal{M}}} \sup_{\mathbf{v}_N \in X_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_X \|q_N\|_M} \geq C(N)$$

The spurious modes set

$$Z_{N,M} = \{q_M \in \mathcal{P}_M, \ \ b_N(q_M, \mathbf{v}_N) = 0, \forall \mathbf{v}_N \in X_N\}.$$

- The dimension of $Z_{N,N}$ is 7 for $d = 2$ and $12N + 3$ for $d = 3$ when $M = N$.
- The dimension of $Z_{N,M}$ is zero when
  - $M = N - 2$
  - $\frac{M}{N} \leq \tau < 1$
- The inf-sup condition decays to zero like $N^{\frac{1-d}{2}}$ when $M = N - 2$
- A uniform inf-sup condition of the family $(X_N, \mathcal{P}_M)$ when $\frac{M}{N} \leq \tau < 1$

Numerical implementation

The equivalent matrix formulation of the Stokes problem is

$$\begin{aligned} \mathbf{A_N}\underline{\mathbf{U}}_N + \mathbf{D_M}\underline{p}_M &= \underline{\mathbf{f}}_N \\ \mathbf{D_M}^T\underline{\mathbf{U}}_N &= 0. \end{aligned}$$

Where

- $\underline{\mathbf{U}}_N$ is a vector of unknowns for $\mathbf{u}_N$
- $\underline{p}_M$ is a vector of unknowns for p_M
- $\underline{\mathbf{f}}_N$ is a vector of the data $\mathbf{f}_N$
- $\mathbf{A}_N$ is the discrete Laplace operator
- $\mathbf{D}_M$ is the discrete Gradient operator.

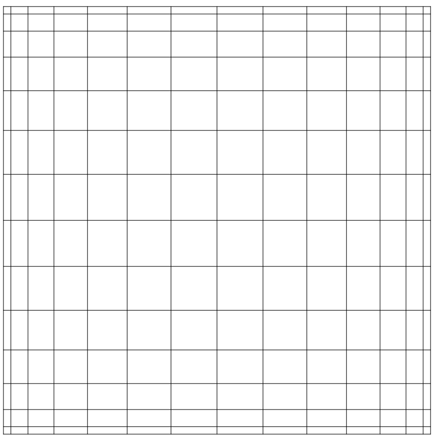
- $\mathbf{u}_N$ is represented at the GLL nodes by :

$$u_N^r(x,y) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} u_N^r(\xi_i,\xi_j) h_i(x) h_j(y).$$

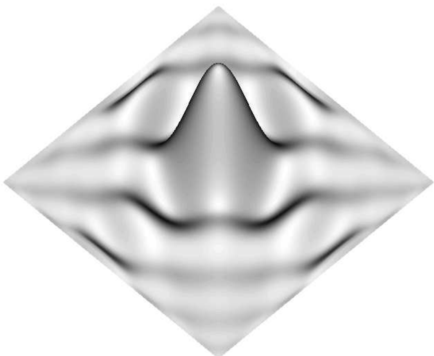
and

- p_M is represented at the G.L. nodes (or Internal GLL) by :

$$p_M(x,y) = \sum_{i=0}^{M} \sum_{\substack{j=0 \\ i+j\neq 0}}^{M} p_M(\zeta_i,\zeta_j) \ell_i(x) \ell_j(y).$$



grille



$h_i \times h_j$

Uzawa algorithm

The pressure can be solved directly from

$$(\mathbf{D}_M^T \mathbf{A}_N^{-1} \mathbf{D}_M)\underline{p}_M = (\mathbf{D}_M^T \mathbf{A}_N^{-1} \mathbf{B}_N)\underline{\mathbf{f}}_N.$$

The Uzawa matrix

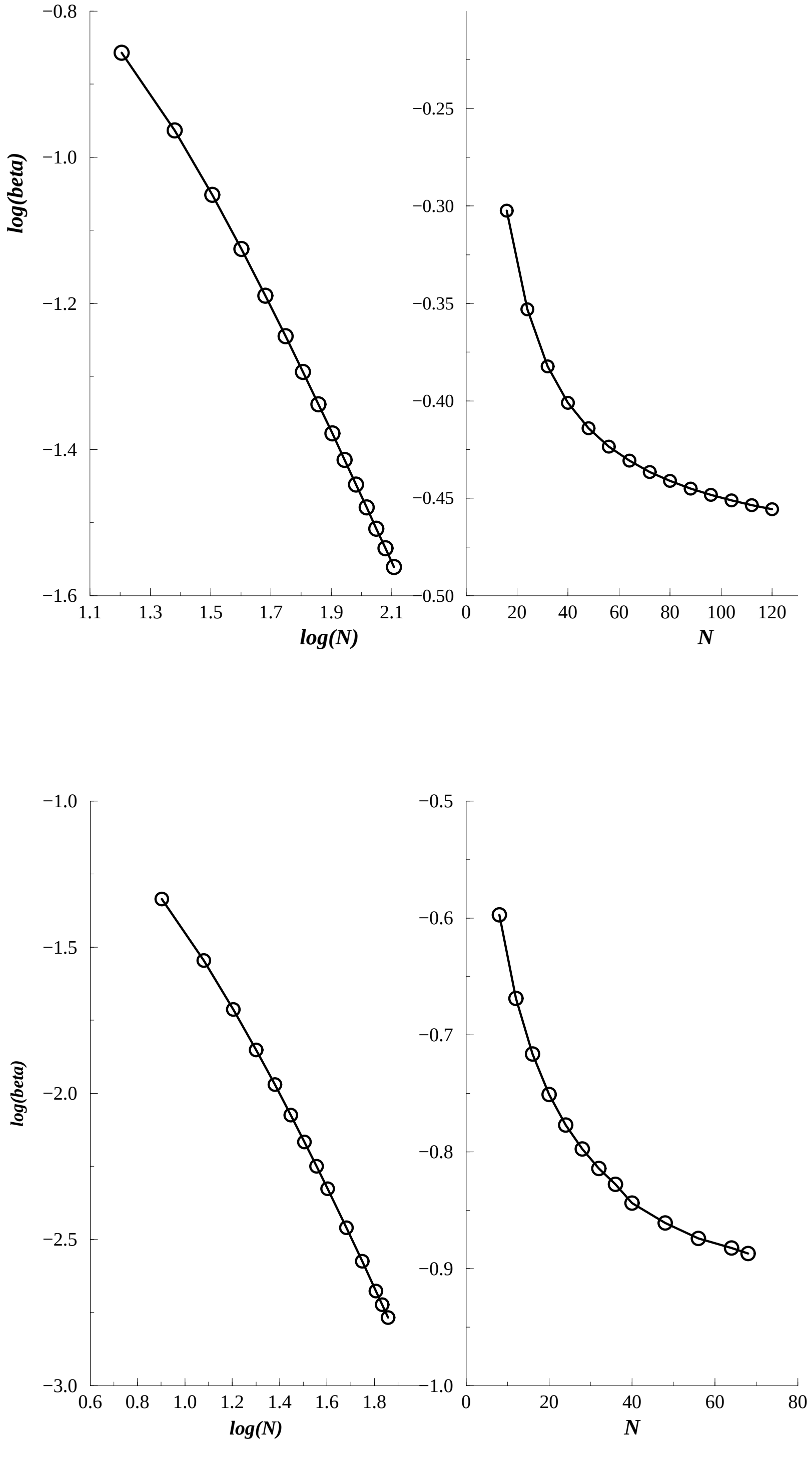
$$\mathbf{S}_{N,M} = (\mathbf{D}_M^T \mathbf{A}_N^{-1} \mathbf{D}_M)$$

is

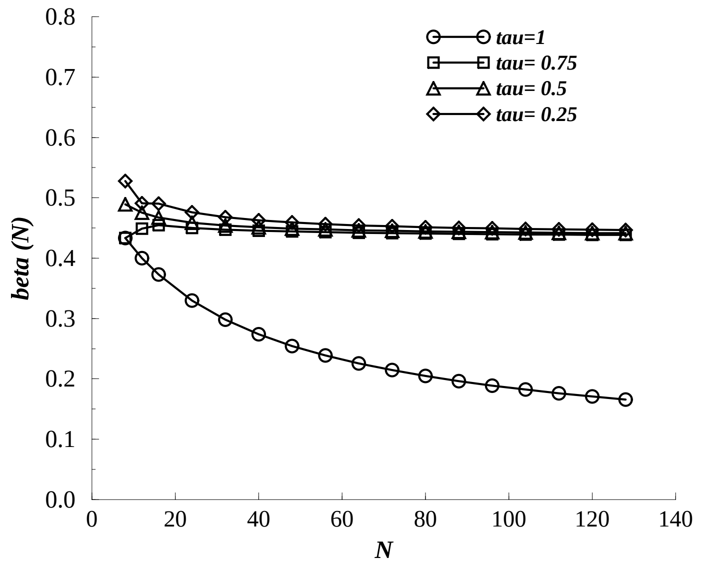
- of dimension $(M+1)^d$,
- full, symmetric and positive definite,
- The condition number κ_τ and the inf-sup constant β_τ of $\mathbf{B}_M^{-1}\mathbf{S}_{N,M}$ verify :

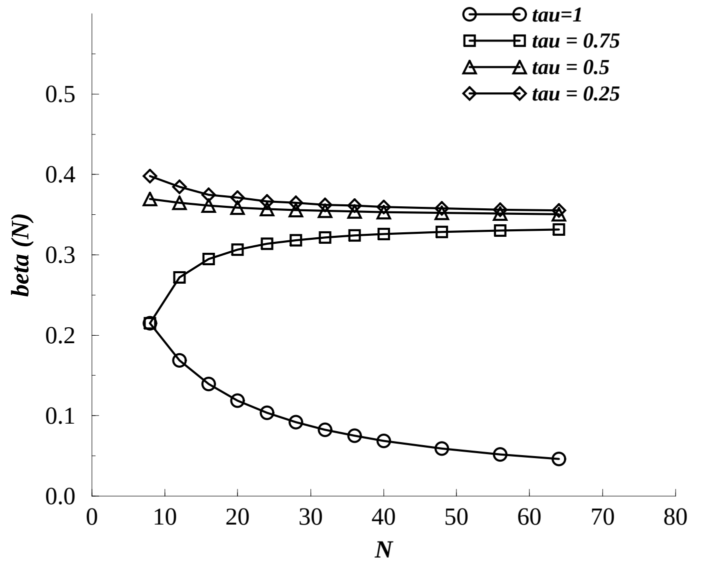
$$\kappa_\tau = \frac{C}{\lambda_{\min}},$$
$$\beta_\tau \approx C\sqrt{\lambda_{\min}(\mathbf{B}_M^{-1}\mathbf{S}_{N,M})}.$$

- P.C.G with the mass matrix $\mathbf{B}_M$ as a preconditionner is applied.

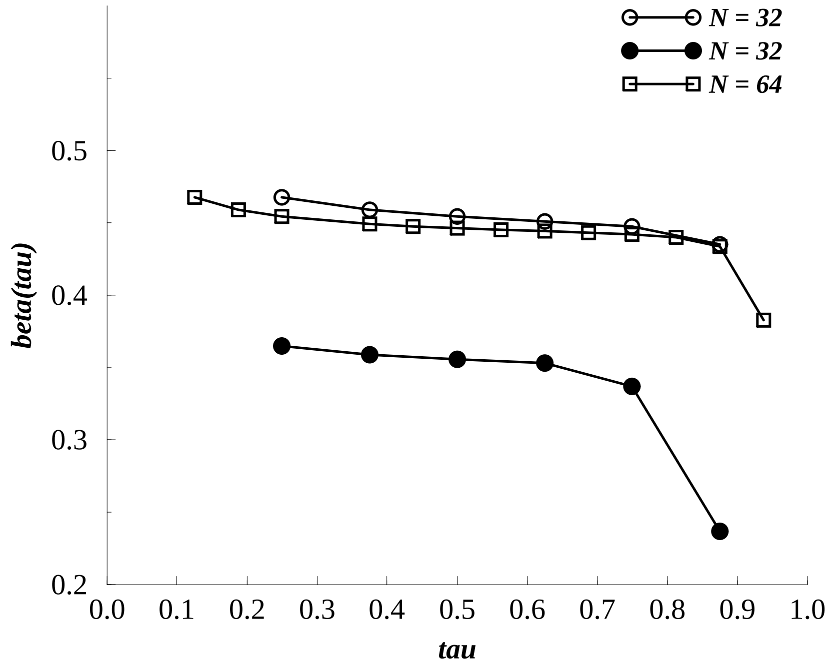


The $\beta_1(N)$ behaviour v.s N in 2 and 3 D

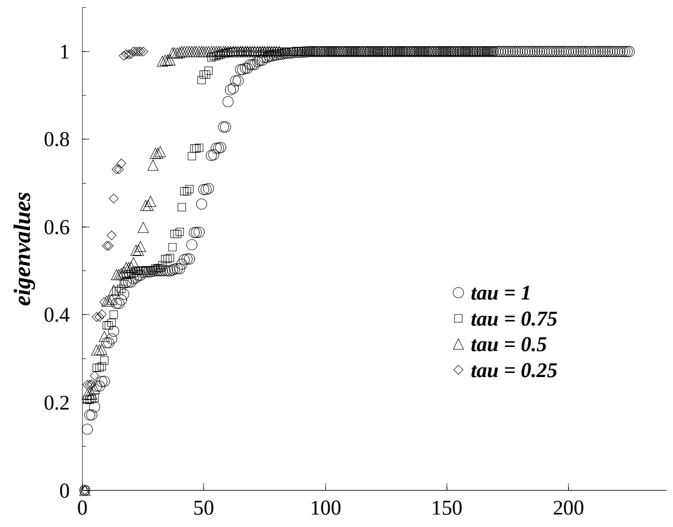


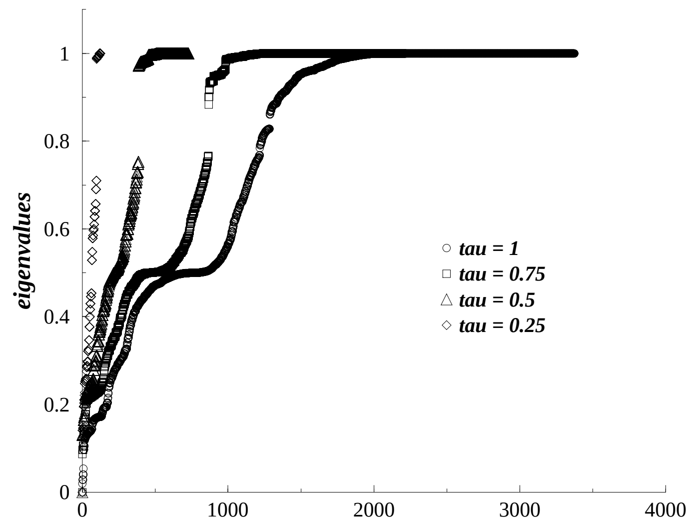


The $\beta_1(N)$ behaviour v.s N,
for different values of τ in 2 and 3 D

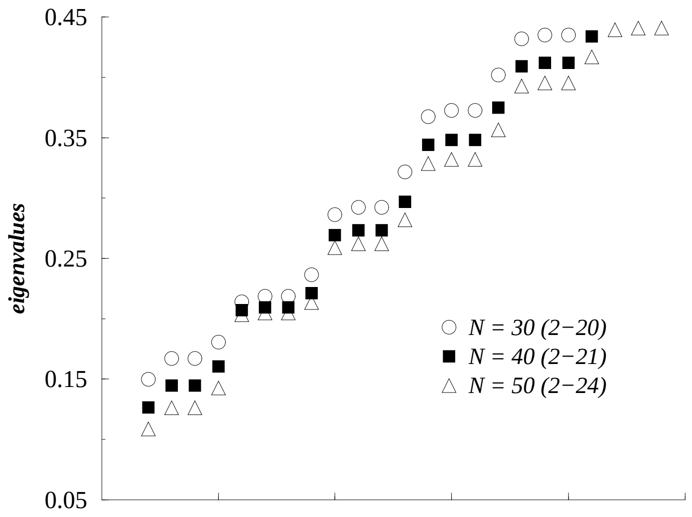


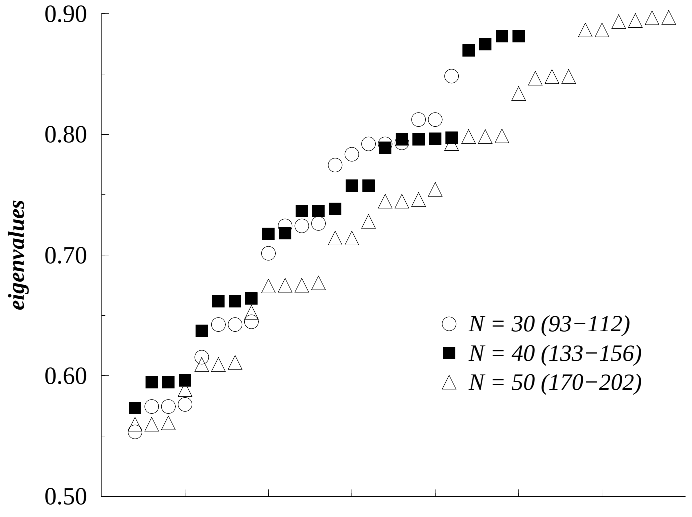
The inf-sup constant (β_τ) behaviour with respect to τ.



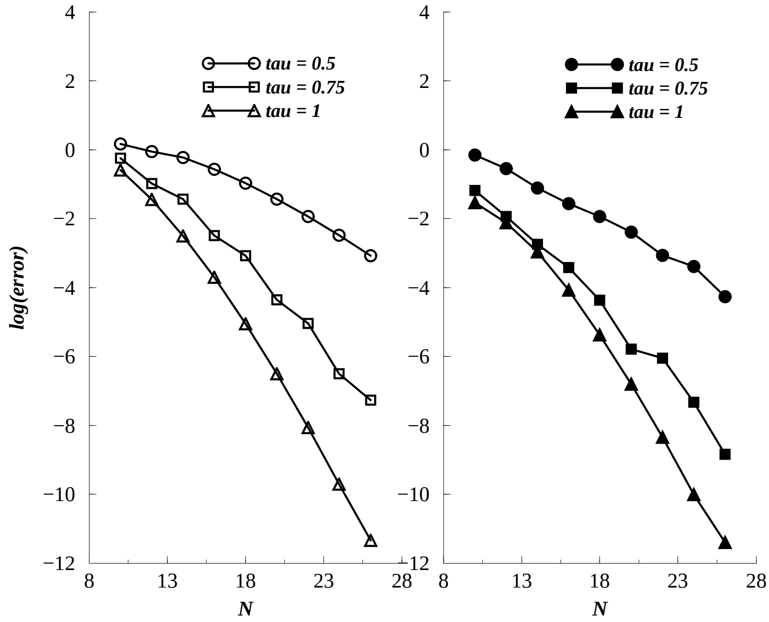


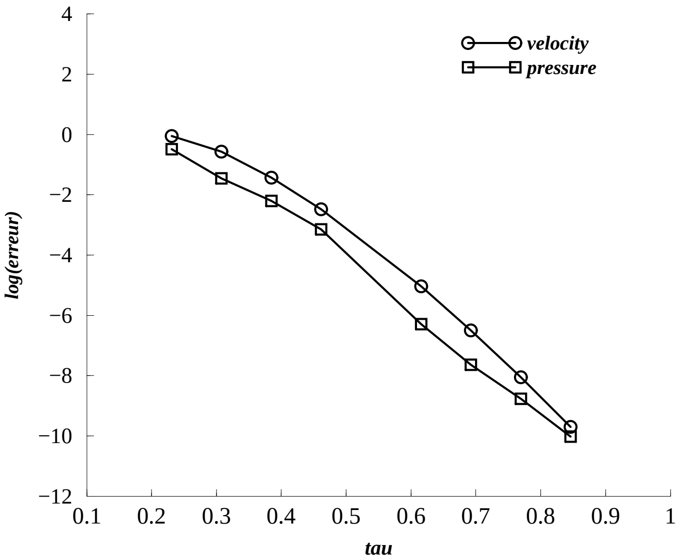
Spectra of the Uzawa's operator in 2 and 3 dimensions



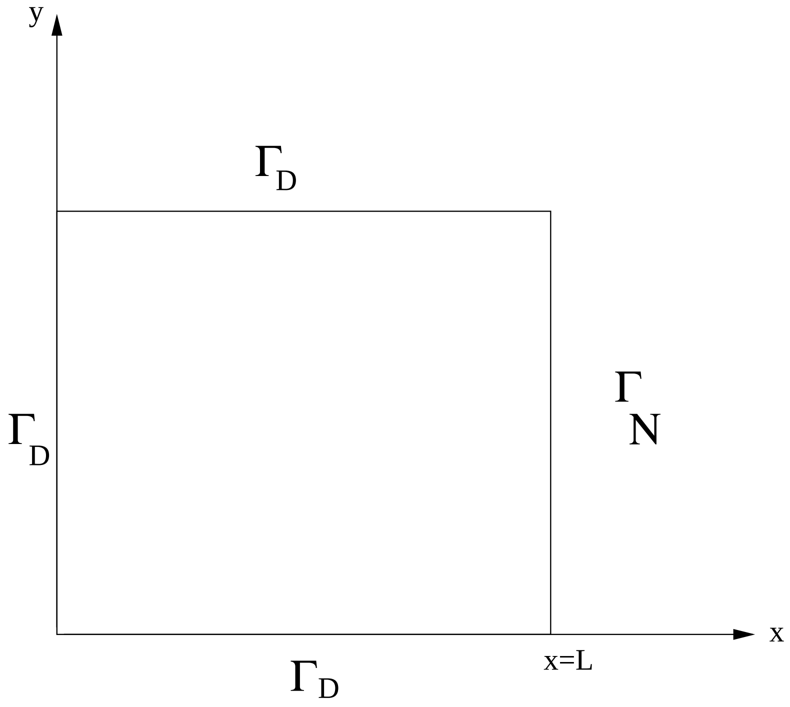


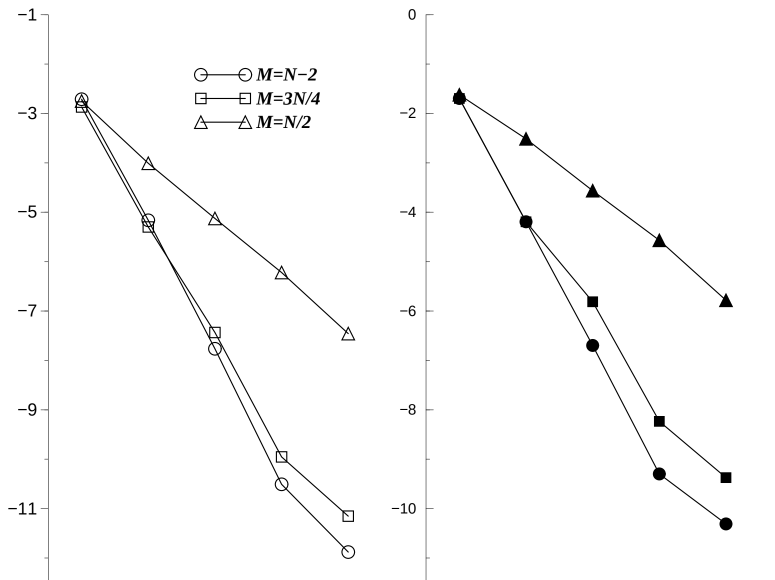
Some eigenvalues of Uzawa's spectra





Logarithm of the errors as a function of N





Logarithm of the errors as a function of N for L-shaped domain

The extact solution is :

$$\mathbf{u} = \begin{pmatrix} \pi e^{\pi y}\cos(\pi x) + \sin(\pi y) \\ \pi e^{\pi y}\sin(\pi x) + \cos(\pi x) \end{pmatrix}$$

$$p = \sin(\pi(x+y))$$

For different values of N and the number of iterations required by the iterative procedure to reach a given accuracy (here 10^{-8}) is

N	20	25	30	35	40	45	45	50	61
2D	16	16	16	17	17	17	17	17	17
3D	23	24	24	25	26	26	26	26	26

The semi-discrete Stokes problem O(1)

$$
\begin{aligned}
\frac{\mathbf{u}^{m+1}-\mathbf{u}^{m}}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{m+1} + \nabla p^{m+1} &= \mathbf{f}^{m+1} \ \text{ in } \Omega,\\
\text{div } \mathbf{u}^{m+1} &= 0 \quad \text{ in } \Omega,\\
\mathbf{u}^{m+1} &= 0 \ \text{ on } \partial\Omega.
\end{aligned}
$$

The Weak formulation reads as : find
$(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ s. t.

$$
\begin{aligned}
A(\mathbf{u}^{m+1}, \mathbf{v}) + b(\mathbf{v}, p^{m+1}) &= L(\mathbf{v}) \quad \forall \mathbf{v},\\
b(\mathbf{u}^{m+1}, q) &= 0 \qquad \forall q,
\end{aligned}
$$

where

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{v}) &= \int_\Omega \frac{\mathbf{u}-\mathbf{u}^m}{\Delta t}\mathbf{v}\, d\mathbf{x} + \frac{1}{Re}\int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\, d\mathbf{x}\\
b(\mathbf{v}, q) &= -\int_\Omega q(\text{div } \mathbf{v})\, d\mathbf{x}\\
L(\mathbf{v}) &= \int_\Omega \mathbf{f}^{m+1}\mathbf{v}\, d\mathbf{x}
\end{aligned}
$$

Mixed $P_N \times P_M$ spectral element applied to the unsteady Stokes problem

For any real number $\tau \in]0,1[$ and any couple of integer (N, M) such that $\frac{M}{N} \leq \min(\tau, 1-\frac{2}{N})$ we define

- $X_N = (I\!\!P_N(\Omega))^d (H_0^1(\Omega))^d$,
- $\mathcal{P}_M = I\!\!P_M(\Omega) \cap L_0^2(\Omega)$.

Find $\mathbf{u}_N \in X_N$ and $p_M \in \mathcal{P}_M$ s. t. :

$$\begin{aligned} A_N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_M) &= L_N(\mathbf{v}_N) \quad && \forall \mathbf{v}_N, \\ b_N(\mathbf{u}_N, q_M) &= 0 && \forall q_M, \end{aligned}$$

where

$$\begin{aligned} A_N(\mathbf{u}, \mathbf{v}) &= (\frac{\mathbf{u} - \mathbf{u}^m}{\Delta t}, \mathbf{v})_N + \frac{1}{Re}(\nabla \mathbf{u}, \nabla \mathbf{v})_N \\ b_N(\mathbf{v}, q) &= -(q, \mathrm{div}\ \mathbf{v})_N \\ L_N(\mathbf{v}) &= \left(\mathbf{f}^{m+1}, \mathbf{v}\right)_N \end{aligned}$$

Numerical implementation

We define :

- $X_N = (\mathbb{P}_N(\Omega))^d \cap (H_0^1(\Omega))^d$,
- $\mathcal{P}_M = \mathbb{P}_M(\Omega) \cap L_0^2(\Omega)$.

The equivalent matrix is

$$\begin{aligned} [\mathbf{B}_N + (\frac{\Delta t}{Re})\mathbf{A}_N]\underline{\mathbf{u}}_N^{m+1} + \Delta t \mathbf{D}_M \underline{p}_M^{m+1} &= \\ \mathbf{B}_N[\underline{\mathbf{u}}_N^m + \Delta t \underline{\mathbf{f}}_N^{m+1}], & \\ -\mathbf{D}_M^T \underline{\mathbf{u}}_N^{m+1} &= 0. \end{aligned}$$

The Uzawa matrix is :

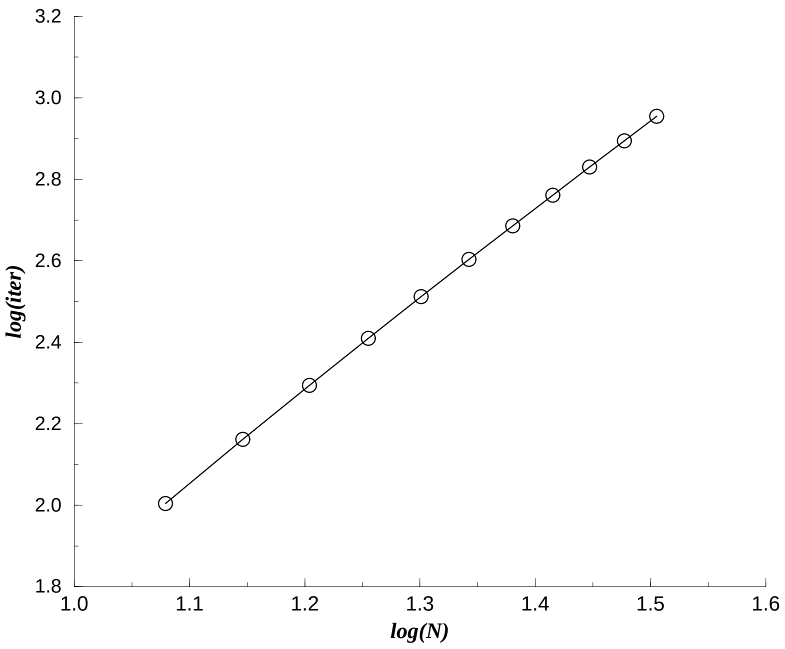
$$\mathbf{S}_{M,N} = \mathbf{D}_M^T [\mathbf{B}_N + (\frac{\Delta t}{Re})\mathbf{A}_N]^{-1} \mathbf{D}_M$$

- The main difficulty of such an approach is that in practical situations, $\frac{\Delta t}{Re} << 1$, so that the matrix is ill conditioned.

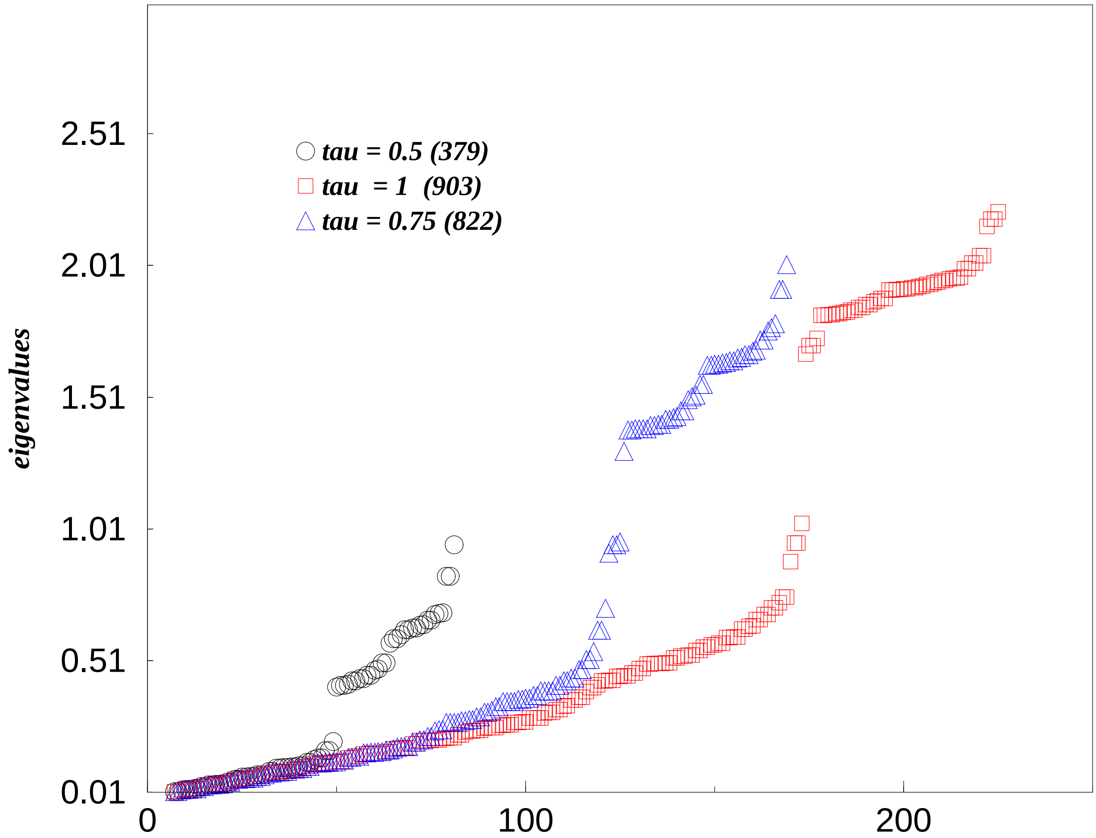
The exact solution is :

$$\begin{aligned} u(x,y) &= \sin(x+5t)\times\sin(y+5t), \\ v(x,y) &= \cos(x+5t)\times\cos(y+5t), \\ p(x,y) &= \sin(x+y+5t). \end{aligned}$$

For different values of N and the number of iterations required by the iterative procedure to reach a given accuracy (here 10^{-8}) is



$\tau = 1$ $Re = 10^3$, and $\Delta t = 0.001$.



Spectra of the Uzawa's operator for N = 16 and for different values of τ, in 2 dimensions. $Re = 10^3$, and $\Delta t = 0.001$.

Second part : Time splitting/spectral element method for the unsteady Stokes problem.

The semi-discrete Stokes problem

$$
\begin{aligned}
\frac{\mathbf{u}^{m+1}-\mathbf{u}^{m}}{\Delta t}-\frac{1}{Re}\Delta\mathbf{u}^{m+1}+\nabla p^{m+1} &= \mathbf{f}^{m+1} \quad \text{in } \Omega,\\
\text{div } \mathbf{u}^{m+1} &= 0 \quad \text{in } \Omega,\\
\mathbf{u}^{m+1} &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

- Chorin-Temam (68)
- Goda (78)
- Van Kan
- Kim Moin
- KIO
- ...

First order Goda scheme.

The semi-discrete Stokes problem

$$
\begin{aligned}
\frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{m+1} + \nabla p^{m+1} &= \mathbf{f}^{m+1} \quad \text{in } \Omega, \\
\text{div } \mathbf{u}^{m+1} &= 0 \quad \text{in } \Omega, \\
\mathbf{u}^{m+1} &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Prediction-diffusion step :

$$
\begin{aligned}
\frac{\mathbf{u}_*^{m+1} - \mathbf{u}^m}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}_*^{m+1} + \nabla p^{m} &= \mathbf{f}^{m+1} \quad \text{in } \Omega, \\
\mathbf{u}_*^{m+1} &= 0. \quad \text{on } \partial\Omega.
\end{aligned}
$$

Correction–projection step :

$$
\begin{aligned}
\frac{\mathbf{u}^{m+1} - \mathbf{u}_*^{m+1}}{\Delta t} + \nabla(p^{m+1} - p^m) &= 0. \quad \text{in } \Omega, \\
\text{div } \mathbf{u}^{m+1} &= 0 \quad \text{in } \Omega, \\
\mathbf{u}^{m+1}.\mathbf{n} &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

First order (N+1) New Goda scheme.

Prediction-diffusion step :

$$\frac{\mathbf{u}_*^{m+1} - \mathbf{u}^m}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}_*^{m+1} + \nabla p^m = \mathbf{f}^{m+1} \qquad \text{in } \Omega,$$

$$\mathbf{u}_*^{m+1} = 0. \qquad \text{on } \partial\Omega.$$

Correction–projection step :

$$\frac{\mathbf{u}^{m+1} - \mathbf{u}_*^{m+1}}{\Delta t} + \nabla \psi = 0. \qquad \text{in } \Omega,$$

$$\text{div } \mathbf{u}^{m+1} = 0 \qquad \text{in } \Omega,$$

$$\mathbf{u}^{m+1}.\mathbf{n} = 0 \qquad \text{on } \partial\Omega.$$

and

$$p^{m+1} = \psi + p^m - \frac{1}{Re}\text{div } \mathbf{u}_*^{m+1}$$

Why $p^{m+1} = \psi + p^m - \frac{1}{Re}\text{div } \mathbf{u}_*^{m+1}$?.

The projection step is :

$$\begin{aligned} \frac{\mathbf{u}^{m+1} - \mathbf{u}_*^{m+1}}{\Delta t} + \nabla\psi &= 0. && \text{in } \Omega, \\ \text{div } \mathbf{u}^{m+1} &= 0 && \text{in } \Omega, \\ \mathbf{u}^{m+1}.\mathbf{n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

and

$$\mathbf{u}^{m+1}.\tau = -\Delta t \frac{\partial \psi}{\partial \tau} \qquad \text{on } \partial\Omega.$$

The equivalent problem on the pressure is :

$$\begin{aligned} \Delta\psi &= \frac{\text{div } \mathbf{u}_*^{m+1}}{\Delta t} && \text{in } \Omega, \\ \frac{\partial \psi}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Some remarks (suite).

The Predicted velocity can be read us :

$$\mathbf{u}_*^{m+1} = \mathbf{u}^{m+1} + \Delta t \nabla \psi,$$

$$\text{and } \frac{1}{Re}\Delta \mathbf{u}_*^{m+1} = \frac{1}{Re}\Delta \mathbf{u}^{m+1} + \frac{1}{Re}\Delta t \Delta \nabla \psi,$$

Then the Goda predicted step becomes :

$$\frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{m+1} + \nabla(\psi + p^m - \frac{1}{Re}\Delta t \Delta \psi) = \mathbf{f}$$

$$\text{div } \mathbf{u}^{m+1} = 0$$

$$\mathbf{u}^{m+1}.\mathbf{n} = 0, \qquad \mathbf{u}^{m+1}.\tau = -\Delta t \frac{\partial \psi}{\partial \tau}.$$

Using now :

$$\Delta t \; \Delta \psi = \text{div } \mathbf{u}_*^{m+1} \qquad \text{in } \Omega,$$

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega.$$

Some remarks (suite).

$$\frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} - \frac{1}{Re}\Delta \mathbf{u}^{m+1} +$$

$$\nabla\left(\psi + p^m - \frac{1}{Re}\text{div } \mathbf{u}_*^{m+1}\right) = \mathbf{f}$$

$$\text{div } \mathbf{u}^{m+1} = 0 \qquad \text{in } \Omega,$$

$$\mathbf{u}^{m+1} = 0 \qquad \text{on } \partial\Omega.$$

Equivalence between semi-descret Stokes problem and Goda Scheme is ensured because

$$p^{m+1} := \psi + p^m - \frac{1}{Re}\text{div } \mathbf{u}_*^{m+1}$$

How to solve the projection step ?

The problem to solve can be written as a Darcy problem :

$$\begin{aligned} \mathbf{u} + \operatorname{grad} p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{on} \quad \Omega, \\ \mathbf{u} \,.\, \mathbf{n} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Poisson-Neumann formulation : find $(\mathbf{u}, p)$ in $L^2(\Omega)^2 \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big)$ such that

$$\begin{aligned} (\operatorname{grad} p \,, \operatorname{grad} q\,) &= (\mathbf{f}\,, \operatorname{grad} q\,)\,, \forall q. \\ \mathbf{u} &= \mathbf{f} - \operatorname{grad} p. \end{aligned}$$

Mixed formulation : find $(\mathbf{u}, p)$ in $H_0(\operatorname{div}\,, \Omega) \times L^2_0(\Omega)$ such that

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) - (p, \operatorname{div} \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\operatorname{div}\,, \Omega), \\ -(q, \operatorname{div} \mathbf{u}) &= 0 \qquad\qquad \forall q \in L^2_0(\Omega). \end{aligned}$$

Non stable Spectral element for the projection step

Discrete Poisson-Neumann formulation (PN)

- $X_N = (P_N(\Omega))^2$
- $M_N = P_N(\Omega) \cap L_0^2(\Omega)$

Find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that:

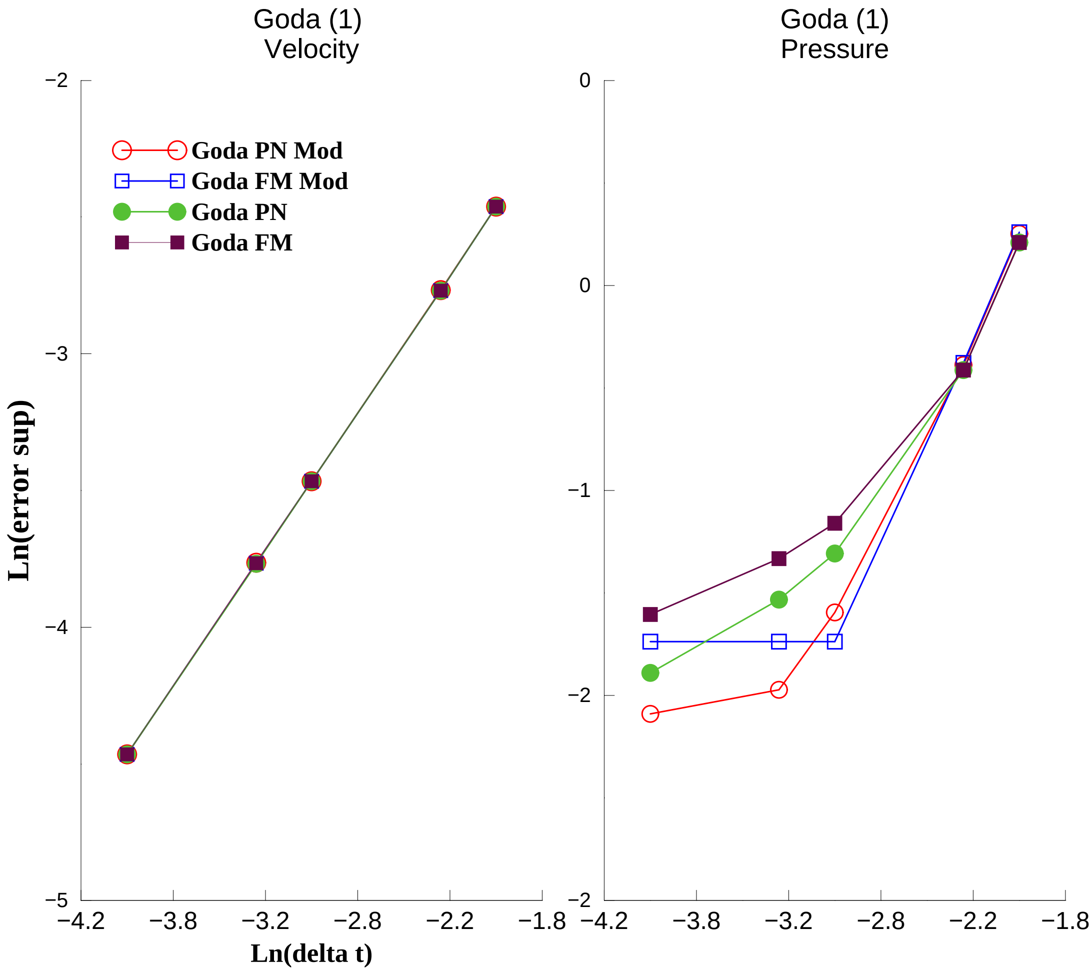
$$\begin{aligned} (\text{grad}\, p_N\ , \text{grad}\, q_N\ )_N &= (\mathbf{f}_N\ , \text{grad}\, q_N\ )\,, \forall q_N \in M_N \\ \mathbf{u}_N &= \mathbf{f}_N - \text{grad}\, p_N\,. \end{aligned}$$

Mixed formulation (FM1)

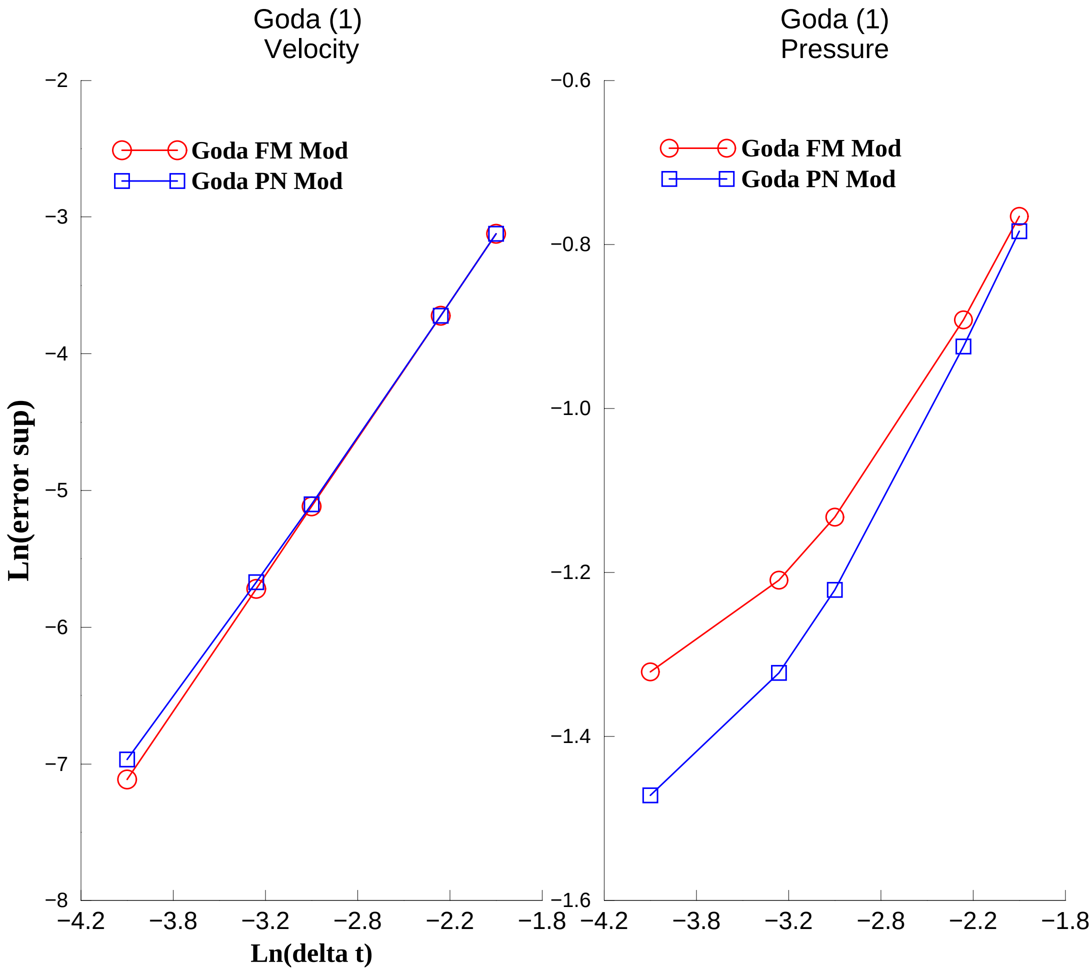
- $X_N = (P_N(\Omega))^2 \cap H_0(\text{div}\ , \Omega)$
- $M_N = P_N(\Omega) \cap L_0^2(\Omega) \backslash$ spurious modes.

Find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that:

$$\begin{aligned} (\mathbf{u}_N, \mathbf{v}_N)_N - (p_N, \text{div}\ \mathbf{v}_N) &= (\mathbf{f}_N, \mathbf{v}_N)_N \ \forall \mathbf{v}_N, \\ -(q_N, \text{div}\ \mathbf{u}_N) &= 0 \qquad\qquad \forall q_N\,. \end{aligned}$$



$\log_{10}$ of the errors as a function of $\log_{10}$ of Δt



$\log_{10}$ of the errors as a function of $\log_{10}$ of Δt

Stable spectral element for the projection step :

Discrete Poisson-Neumann formulation (PN)

- $X_N = (P_N(\Omega))^2$
- $M_N = P_{N-2}(\Omega) \cap L_0^2(\Omega)$

Find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that:

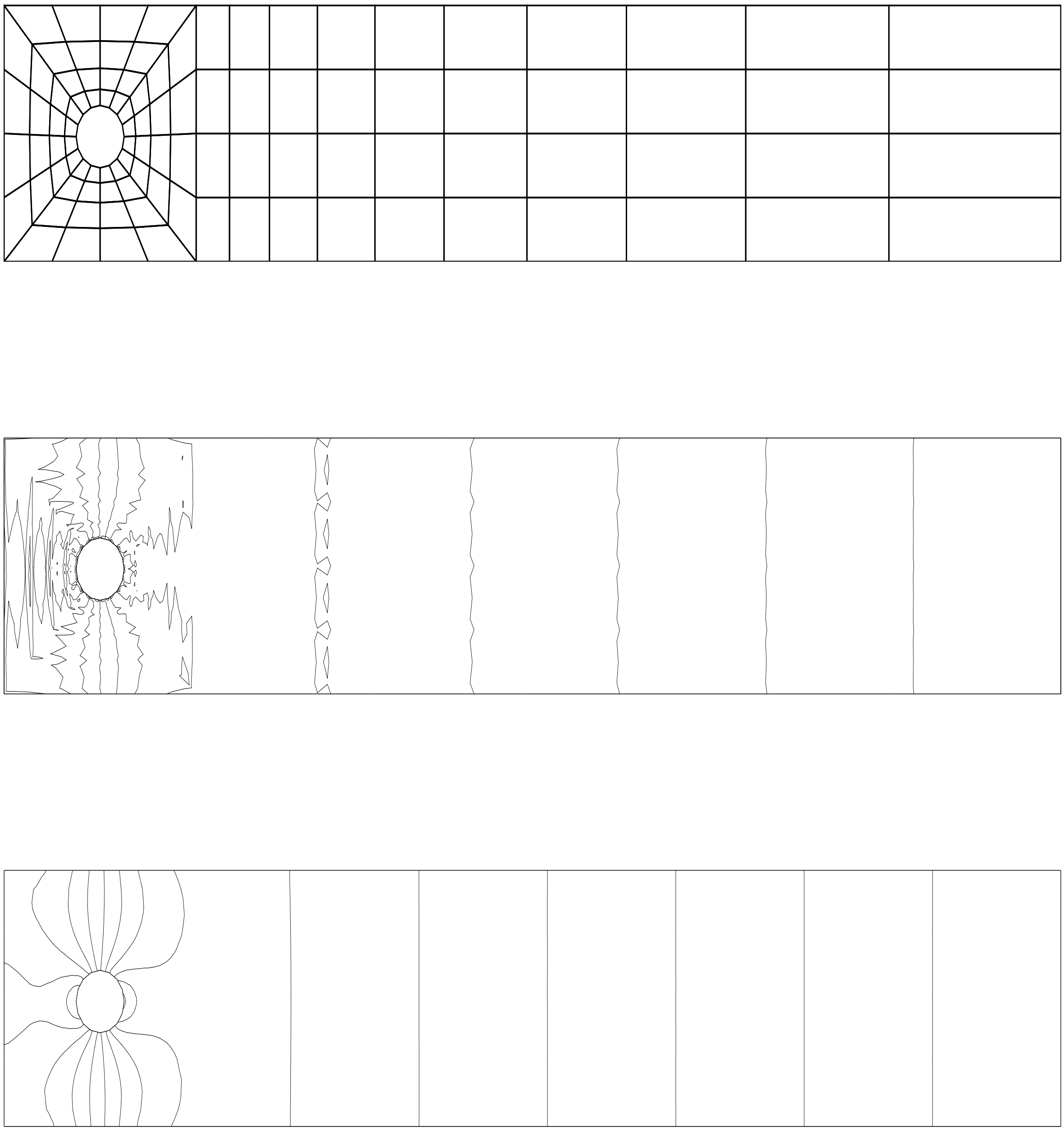
$$\begin{aligned} (\text{grad}\, p_N \,, \text{grad}\, q_N\,)_N &= (\mathbf{f}_N \,, \text{grad}\, q_N\,)\,, \forall q_N \in M_N \\ \mathbf{u}_N &= \mathbf{f}_N - \text{grad}\, p_N. \end{aligned}$$

Mixed formulation (FM1)

- $X_N = (P_N(\Omega))^2 \cap H_0(\text{div}\,, \Omega)$
- $M_N = P_{N-2}(\Omega) \cap L_0^2(\Omega)$

Find $(\mathbf{u}_N, p_N)$ in $X_N \times M_N$ such that:

$$\begin{aligned} (\mathbf{u}_N, \mathbf{v}_N)_N - (p_N, \text{div}\, \mathbf{v}_N) &= (\mathbf{f}_N, \mathbf{v}_N)_N \; \forall \mathbf{v}_N, \\ -(q_N, \text{div}\, \mathbf{u}_N) &= 0 \qquad\qquad \forall q_N. \end{aligned}$$



A Stokes spectral element for the projection step (FM2)

We consider the algebraic system :

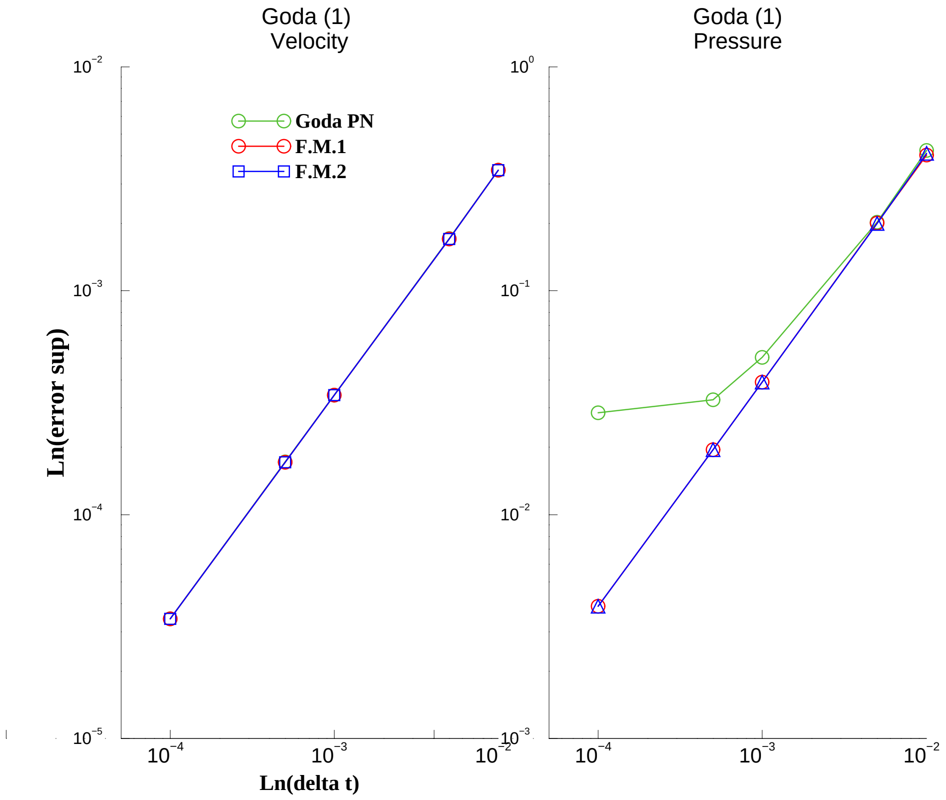
$$\begin{aligned}
[\mathbf{B}_N + (\frac{\Delta t}{Re})\mathbf{A}_N]\underline{\mathbf{u}}_N^{m+1} + \Delta t \mathbf{D}_M \underline{p}_M^{m+1} &= \\
\mathbf{B}_N[\underline{\mathbf{u}}_N^{m} + \Delta t \underline{\mathbf{f}}_N^{m+1}], \\
-\mathbf{D}_M^T \underline{\mathbf{u}}_N^{m+1} &= 0.
\end{aligned}$$

Than the projection step becomes :

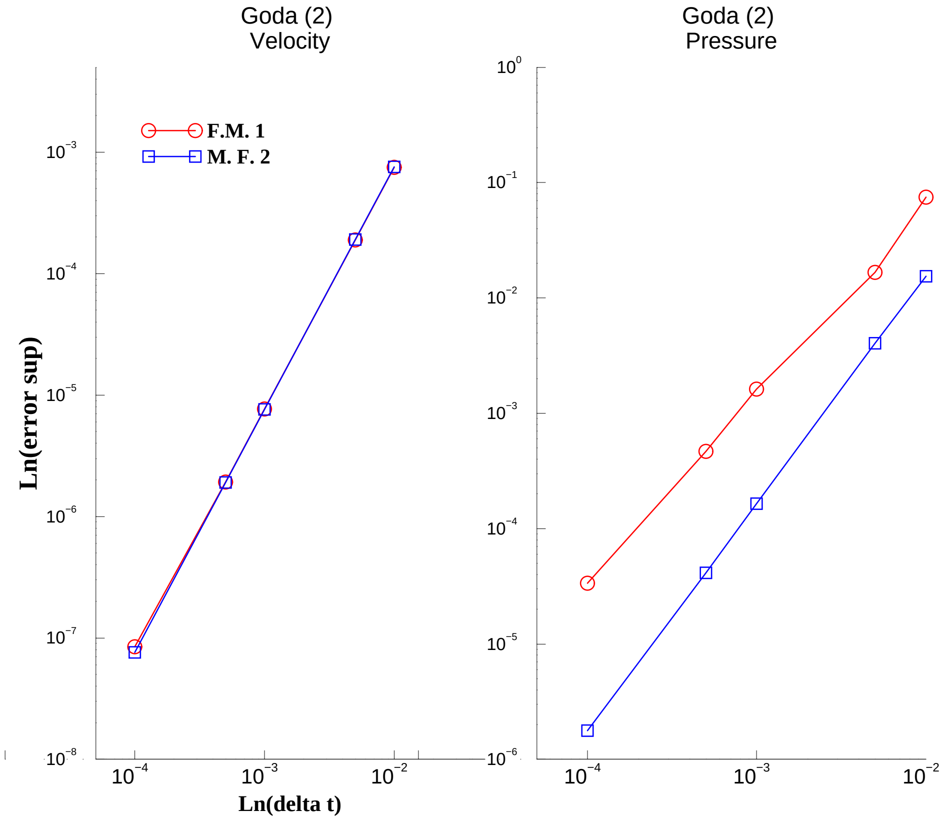
$$\begin{aligned}
\mathbf{B}_N \left(\underline{\mathbf{u}}_N^{m+1} - \underline{\mathbf{u}}_N^{*}\right) + \Delta t \mathbf{D}_M \underline{\Psi} &= 0. \\
-\mathbf{D}_M^T \underline{\mathbf{u}}_N^{m+1} &= 0.
\end{aligned}$$

Wich is a stable discretisation of the problem : find $(\mathbf{u}, p)$ in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

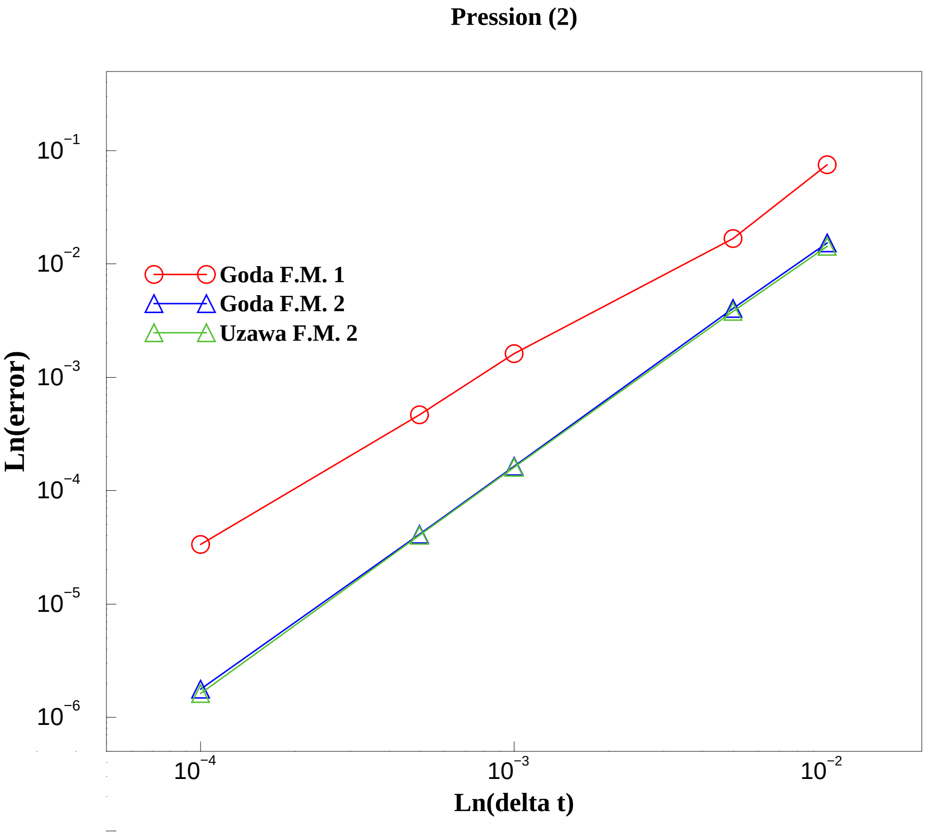
$$\begin{aligned}
(\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\ \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^2, \\
-(q, \mathrm{div}\ \mathbf{u}) &= 0 \qquad\qquad \forall q \in L_0^2(\Omega).
\end{aligned}$$



$\log_{10}$ of the errors as a function of $\log_{10}$ of Δt



$\log_{10}$ of the errors as a function of $\log_{10}$ of Δt



$\log_{10}$ of the errors as a function of $\log_{10}$ of Δt

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