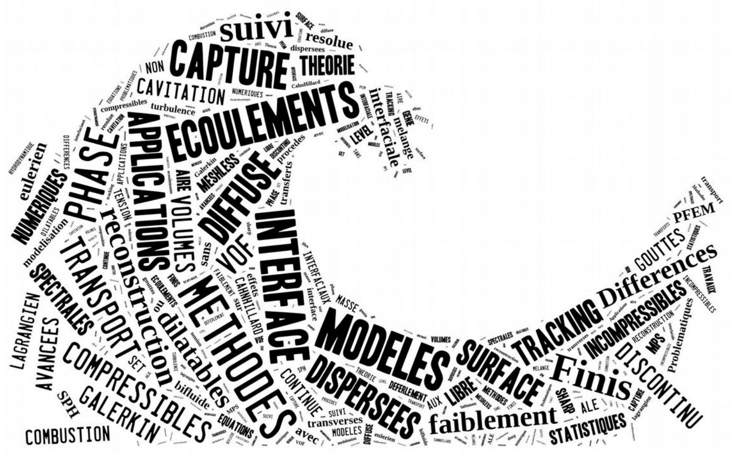
14ème ECOLE DE PRINTEMPS
de Mécanique des Fluides Numérique



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Finite Difference and Finite Volume Methods for Incompressible Flows

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Governing Equations, Finite Difference and Finite Volume Methods

Mass Continuity Equation

Mass Conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Expanding the divergence operator in Cartesian coordinates,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

$\rho = density, t = time$

$\nabla\ is\ a\ vector\ operator$: $\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$\vec{u}\ is\ a\ vector$: $u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$

Momentum and Scalar Transport Equations

$$\frac{\partial(\rho u_x)}{\partial t}+\frac{\partial}{\partial x}(\rho u_x u_x)+\frac{\partial}{\partial y}(\rho u_y u_x)+\frac{\partial}{\partial z}(\rho u_z u_x)=$$
$$\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}-\frac{\partial p}{\partial x}+\rho g_x$$

$$\frac{\partial(\rho u_y)}{\partial t}+\frac{\partial}{\partial x}(\rho u_y u_x)+\frac{\partial}{\partial y}(\rho u_y u_y)+\frac{\partial}{\partial z}(\rho u_z u_y)=$$
$$\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}-\frac{\partial p}{\partial y}+\rho g_y$$

Momentum and Scalar Transport Equations

$$\frac{\partial(\rho u_z)}{\partial t} + \frac{\partial}{\partial x}(\rho u_z u_x) + \frac{\partial}{\partial y}(\rho u_y u_z) + \frac{\partial}{\partial z}(\rho u_z u_z) = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g_z$$

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho u_x \phi) + \frac{\partial}{\partial y}(\rho u_y \phi) + \frac{\partial}{\partial z}(\rho u_z \phi) = \frac{\partial}{\partial x}\left(\Gamma_\phi \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma_\phi \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma_\phi \frac{\partial \phi}{\partial z}\right) + S_\phi$$

Γ_ϕ is an "exchange" coefficient for that scalar and S_ϕ is the source term.

Momentum and Scalar Transport Equations

τ is a tensor with 9 components; It is however symmetric. For a Newtonian fluid,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

μ = μ(T), a property of the fluid

$$\tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \text{etc.}$$

For non-Newtonian fluids, the viscosity is a function of the strain rate. Toothpaste, gels, paints, etc. are examples non-Newtonian fluids.

Simplifications

Steady flow:

$$\frac{\partial}{\partial t}(\quad) = 0$$

Constant density flows

$$\rho = constant$$

One-dimensional flow

$$\frac{\partial(\rho u_x)}{\partial t} + \frac{\partial}{\partial x}(\rho u_x u_x) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \rho g_x$$

Simplifications

Two-dimensional flows

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) =$$

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g_x$$

Similarly in y direction

Taylor Series Expansions

Consider a continuous function f. Let us assume that all the derivatives (to the degree of interest) are also finite. Then

$$f(x_o + \Delta x) = f(x_o) + \Delta x \left.\frac{df}{dx}\right|_{x_o} + \frac{\Delta x^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x_o} + \frac{\Delta x^3}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x_o} + \ldots$$

$$f(x_o - \Delta x) = f(x_o) - \Delta x \left.\frac{df}{dx}\right|_{x_o} + \frac{\Delta x^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x_o} - \frac{\Delta x^3}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x_o} + \ldots$$

Taylor Series Expansions

We can subtract (2) from (1) and write:

$$\frac{f(x_o+\Delta x)-f(x_o-\Delta x)}{2\Delta x}=\left.\frac{df}{dx}\right|_{x_o}+\left.\frac{\Delta x^2}{6}\frac{d^3f}{dx^3}\right|_{x_o}+\ \dots$$

This series can be truncated by dropping all derivatives higher than $\frac{df}{dx}$.

$$\left.\frac{df}{dx}\right|_{x_o}\approx\frac{f(x_o+\Delta x)-f(x_o-\Delta x)}{2\Delta x}$$

Taylor Series Expansions

Now add the two Taylor Expansions

$$f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x) = \Delta x^2 \left.\frac{d^2 f}{dx^2}\right|_{x_o} + \frac{\Delta x^4}{12} \left.\frac{d^4 f}{dx^4}\right|_{x_o} + \cdots$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x_o} \approx \frac{f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x)}{\Delta x^2}$$

The neglected term lead with $\frac{\Delta x^2}{12} \left.\frac{d^4 f}{dx^4}\right|_{x_o}$

Other stencils

$$f'' = \frac{f(x_o+2\Delta x)+16f(x_o+\Delta x)-30f(x_o)+16f(x_o-\Delta x)-f(x_o-2\Delta x)}{12\Delta x^2} + O(\Delta x^4)$$

$$f' = \frac{-f(x_o+2\Delta x)+8f(x_o+\Delta x)-8f(x_o-\Delta x)+f(x_o-2\Delta x)}{12\Delta x} + O(\Delta x^4)$$

$$f'' = \frac{2f(x_o)-5f(x_o-\Delta x)+4f(x_o-2\Delta x)-f(x_o-3\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

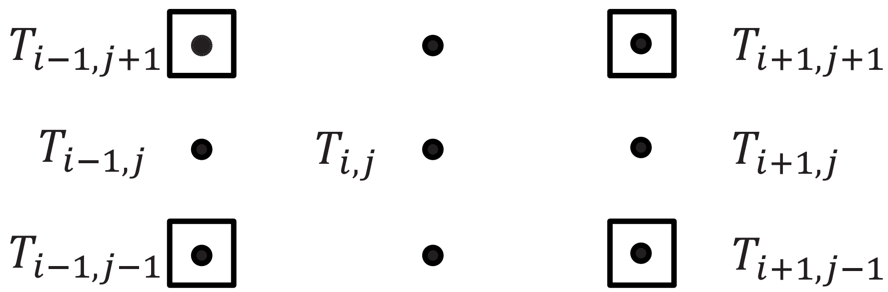
All of above can be derived by above procedure

Mixed Derivative $\frac{\partial^2 T}{\partial x \partial y}$

To Discretize the mixed derivative first write

$$\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) \approx \frac{\left.\frac{\partial T}{\partial y}\right|_{i+1,j} - \left.\frac{\partial T}{\partial y}\right|_{i-1,j}}{2\Delta x}$$

Then



$$\left.\frac{\partial T}{\partial y}\right|_{i+1,j} \approx \frac{T_{i+1,j+1} - T_{i+1,j-1}}{2\Delta y}$$

$$\left.\frac{\partial T}{\partial y}\right|_{i-1,j} \approx \frac{T_{i-1,j+1} - T_{i-1,j-1}}{2\Delta y}$$

FVM

FVM discretizes the PDE by integrating over these discrete control volumes.

$$\iint_{\delta V} \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) dV + \iint_{\delta V} \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) dV = 0$$

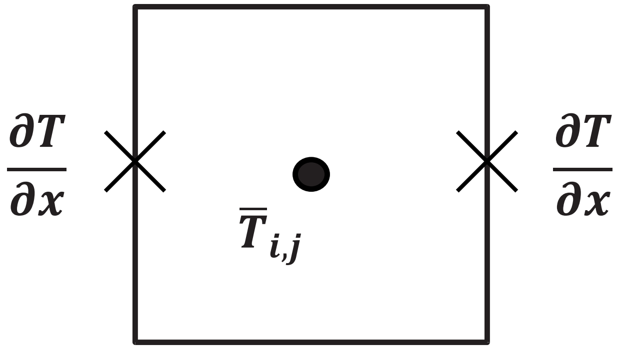
We assume that there is an <u>average</u> value for the control volume.

$$\left.\int_{\Delta y} k\frac{\partial T}{\partial x} dy\right|_{x+} - \left.\int_{\Delta y} k\frac{\partial T}{\partial x} dy\right|_{x-} +$$

$$\left.\int_{\Delta x} k\frac{\partial T}{\partial y} dx\right|_{y+} - \left.\int_{\Delta x} k\frac{\partial T}{\partial y} dx\right|_{y-} = 0$$

FVM

Now we need to evaluate $\int_{\Delta y} k_{x+} \frac{\partial T}{\partial x} dy \Big|_{x+}$**,** $\int_{\Delta y} k_{x-} \frac{\partial T}{\partial x} dy \Big|_{x-}$ **at the cell faces.**



The finite volume expression so far is EXACT. The challenge now however is to evaluate the integral using values computed and stored.

FVM

Midpoint Integration:

Let us assume $\frac{\partial T}{\partial x}$ varies linearly over the Δy of a given control volume.

Then $\int_{\Delta y} k \frac{\partial T}{\partial x} dy \Big|_{x+} = k \left(\frac{\partial T}{\partial x}\right)_{mid} \cdot \Delta y$

The second task is to relate $\frac{\partial T}{\partial x}$ to the $\overline{T}_{i,j}$ calculated at cell centers. Again if we assume a linear variation in x, then

$$\left.\frac{\partial \overline{T}}{\partial x}\right|_{x+} = (\overline{T}_{i+1,j} - \overline{T}_{i,j})/\Delta x$$

FVM

Hence in the context of a <u>linear variation</u>, we get the same stencil as the FDM, if k is constant.

$$\left\{k_{x+}\left(\frac{\overline{T}_{i+1,j}-\overline{T}_{i,j}}{\Delta x}\right)-k_{x-}\left(\frac{\overline{T}_{i,j}-\overline{T}_{i-1,j}}{\Delta x}\right)\right\}\cdot\Delta y+$$
$$\left\{k_{y+}\left(\frac{\overline{T}_{i,j+1}-\overline{T}_{i,j}}{\Delta y}\right)-k_{y-}\left(\frac{\overline{T}_{i,j}-\overline{T}_{i,j-1}}{\Delta y}\right)\right\}\cdot\Delta x=0$$

However, $\overline{T}_{i,j}$ that is calculated is not a point value as in FDM, except that the average is also the midpoint value <u>only</u> when the variation is linear If k is not constant, FVM and FDM have different discrete operators.

FVM

Where $k_{i+\frac{1}{2},j}$, $k_{i-\frac{1}{2},j}$ are evaluated by averaging linearly from neighbor values. Or $k_{i+\frac{1}{2},j}$ can be evaluated using $\overline{T}_{i+\frac{1}{2},j}$ where $\overline{T}_{i+\frac{1}{2},j}$ is averaged.

Hence

$$k_{i+\frac{1}{2},j} = \frac{1}{2}\left(k_{i,j} + k_{i+1,j}\right) \quad \text{or}$$

$$\overline{T}_{i+\frac{1}{2},j} = \frac{1}{2}\left(\overline{T}_{i,j} + \overline{T}_{i+1,j}\right) \quad \text{and}$$

$$k_{i+\frac{1}{2},j} = f\left(\overline{T}_{i+\frac{1}{2},j}\right)$$

Either procedure has errors, hard to say which one gives less error. The second scheme includes more local information.

FVM—Face Values

Consider first the one-dimensional equation

$$u\frac{\partial \Phi}{\partial x} = \Gamma \frac{\partial^2 \Phi}{\partial x^2}$$

Where u is the advection velocity and Γ is the "exchange" coefficient or diffusion coefficient.

Linear and Upwind Interpolation

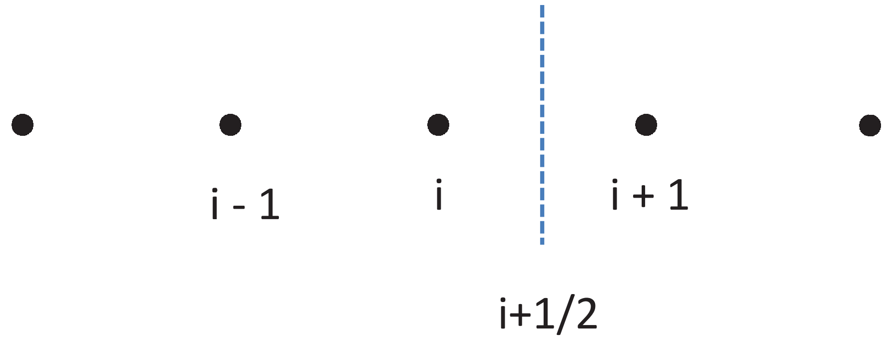
Given $\boldsymbol{\Phi_{i,j}}$ and $\boldsymbol{\Phi_{i+1,j}}$

$$\Phi_{i+\frac{1}{2},j} = \frac{\Delta x_i}{(\Delta x_i + \Delta x_{i+1})}\Phi_{i+1,j} + \frac{\Delta x_{i+1}}{(\Delta x_i + \Delta x_{i+1})}\Phi_{i,j}$$

First order Upwind Interpolation:

$$\Phi_{i+\frac{1}{2},j} = \begin{cases} \Phi_{i,j}, & \mathrm{u}_{i+\frac{1}{2},j} > 0 \\ \Phi_{i+1,j}, & \mathrm{u}_{i+\frac{1}{2},j} < 0 \end{cases}$$

Second-order upwinding



If $u_{i+\frac{1}{2}} > 0,$

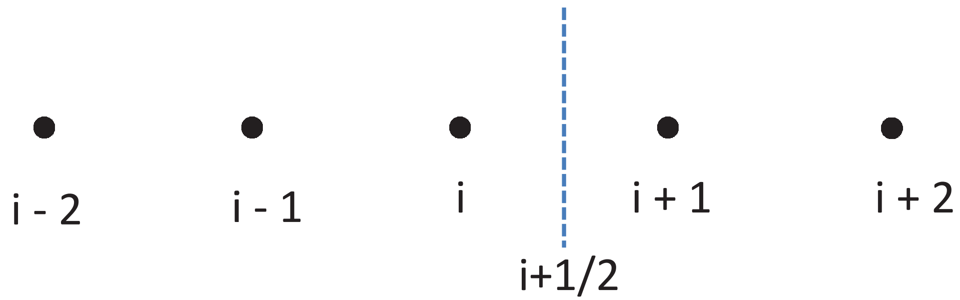
$$\Phi_{i+\frac{1}{2}} = \Phi_i + \frac{\Delta x}{2}\frac{(\Phi_i - \Phi_{i-1})}{\Delta x} = \frac{3}{2}\Phi_i - \frac{1}{2}\Phi_{i-1}$$

If $u_{i+\frac{1}{2}} < 0,$

$$\Phi_{i+\frac{1}{2}} = \Phi_{i+1} + \frac{\Delta x}{2}\frac{(\Phi_{i+1} - \Phi_{i+2})}{\Delta x} = \frac{3}{2}\Phi_{i+1} - \frac{1}{2}\Phi_{i+2}$$

QUICK (Quadratic Upstream Interpolation of Convective Kinematics); Leonard (1981)

Interpolation using both upstream and downstream points



Fit a second-order polynomial between two upstream and one downstream points.

If $u_{i+\frac{1}{2}}$ is > 0

$$\boldsymbol{\Phi}_{i+\frac{1}{2}} = \frac{3\boldsymbol{\Phi}_{i+1} + 6\boldsymbol{\Phi}_{i} - \boldsymbol{\Phi}_{i-1}}{8}$$

Leonard has further developed new schemes call UTOPIA, NIRVANA, ULTIMATE, etc.

Fromm’s method

Fromm’s method can be seen as average of central and second-order upwinding

$$u_{i+\frac{1}{2}} > 0;$$

$$\Phi_{i+\frac{1}{2}} = \frac{(\Phi_i + \Phi_{i+1})}{2} - \frac{1}{4}(\Phi_{i+1} - 2\Phi_i + \Phi_{i-1})$$

$$\mathrm{u}_{i+\frac{1}{2}} < 0$$

$$\Phi_r = \frac{(\Phi_i + \Phi_{i+1})}{2} - \frac{1}{4}(\Phi_{i+2} - 2\Phi_{i+1} + \Phi_i)$$

Issues

- No equation for pressure
- Continuity Equation is a constraint on the momentum velocities
- Pressure has to be computed such that the velocity field is divergence free
- Both steady state and time marching procedures are possible
- Steady state can be obtained by semi-implicit methods

Pressure Poisson Equation

An elliptic partial differential equation can be derived for pressure. When density is constant,

$$\nabla^2 \mathbf{p} = \frac{\partial}{\partial \mathbf{x}}(-\mathbf{C}^{\mathbf{u}} + \mathbf{D}^{\mathbf{u}}) + \frac{\partial}{\partial \mathbf{y}}(-\mathbf{C}^{\mathbf{v}} + \mathbf{D}^{\mathbf{v}})$$

Where $\mathbf{C}^{\mathbf{u}} + \mathbf{D}^{\mathbf{u}}$, $\mathbf{C}^{\mathbf{v}} + \mathbf{D}^{\mathbf{v}}$ are advection and diffusion terms in u and v equations This however requires knowledge of velocity fields that satisfy the divergence free condition. We need an iterative procedure to determine u, v and p. Since the continuity equation is now a constraint, we need to find a procedure to determine p.

Fractional Step Method

The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv)\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

$$\rho\left(\frac{\partial \mathrm{v}}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv)\right) = -\frac{\partial p}{\partial y} + \mu\nabla^2 \mathrm{v}$$

Location of (u, v, p)

Let us consider a finite difference technique with first order accuracy in time and second-order accuracy in space. Let us consider first an algorithm that is explicit in time for advection and diffusion terms. Consider a two-dimensional grid with (i,j) as the nodal indices. Let us locate all three variables (u, v, p) at the same nodal locations. This arrangement is called “colocated” arrangement. This contrasts with “staggered arrangement”.

Discretization

$$\rho \frac{u_{i,j}{}^{n+1} - u_{i,j}{}^{n}}{\Delta t} = H_u{}^n + \frac{p_{i-1,j} - p_{i+1,j}}{2\Delta x}$$

$$\rho \frac{\mathrm{v}_{i,j}{}^{n+1} - \mathrm{v}_{i,j}{}^{n}}{\Delta t} = H_\mathrm{v}{}^n + \frac{p_{i,j-1} - p_{i,j+1}}{2\Delta y}$$

$$H_u{}^n = (-C^u + D^u)^n$$
$$H_\mathrm{v}{}^n = (-C^\mathrm{v} + D^\mathrm{v})^n$$

$$\frac{u_{i+1,j}{}^{n+1} - u_{i-1,j}{}^{n+1}}{2\Delta x} + \frac{\mathrm{v}_{i,j+1}{}^{n+1} - \mathrm{v}_{i,j-1}{}^{n+1}}{2\Delta y} = 0$$

Pressure-Poisson Equation

$$\frac{u_{i+1,j}{}^{n} + \Delta t \frac{1}{\rho}(H_u{}^{n})_{i+1,j} - u_{i-1,j}{}^{n} - \Delta t \frac{1}{\rho}(H_u{}^{n})_{i-1,j}}{2\Delta x}$$

$$+ \frac{v_{i,j+1}{}^{n} + \Delta t \frac{1}{\rho}(H_v{}^{n})_{i,j+1} - v_{i,j-1}{}^{n} - \Delta t \frac{1}{\rho}(H_v{}^{n})_{i,j-1}}{2\Delta y}$$

$$= \Delta t \frac{\left(p_{i+2,j}{}^{n+1} - p_{i,j}{}^{n+1} - p_{i,j}{}^{n+1} - p_{i-2,j}{}^{n+1}\right)}{4\Delta x^2}$$

$$+\Delta t \frac{\left(p_{i,j+2}{}^{n+1} - p_{i,j}{}^{n+1} - p_{i,j}{}^{n+1} - p_{i,j-2}{}^{n+1}\right)}{4\Delta y^2}$$

Simplifying the notation,

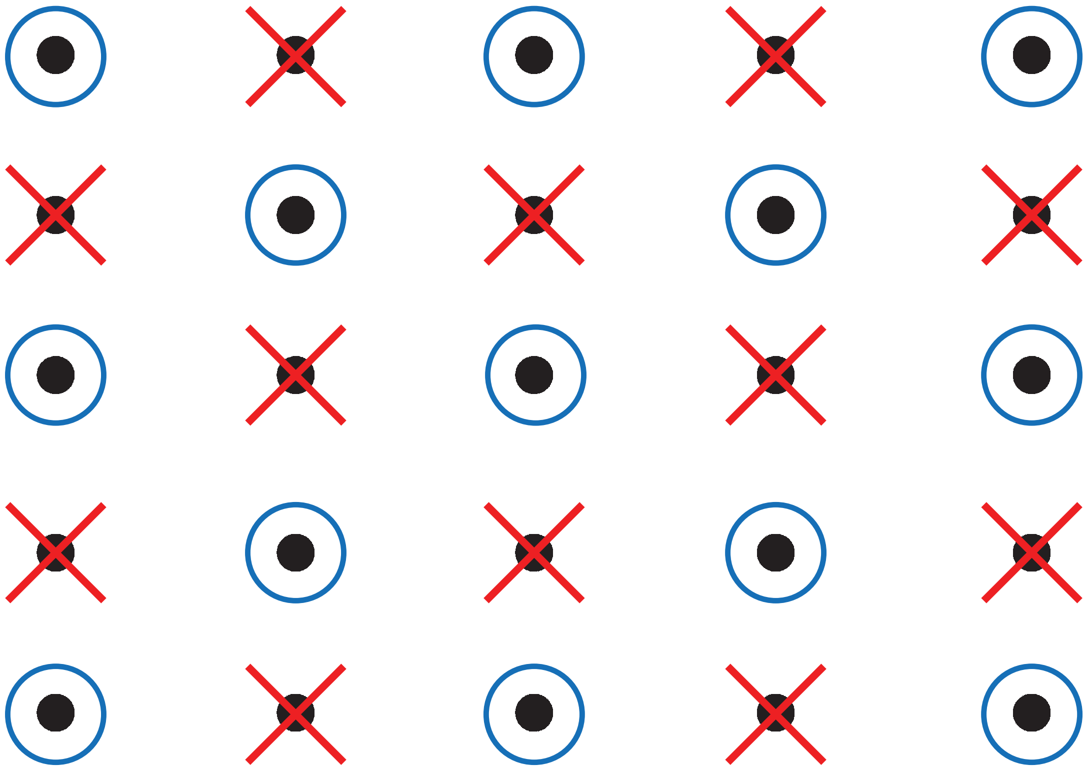
$$\frac{\widehat{u}_{i+1,j} - \widehat{u}_{i-1,j}}{2\Delta x} + \frac{\hat{\mathrm{v}}_{i,j+1} - \hat{\mathrm{v}}_{i,j-1}}{2\Delta y}$$

$$= \frac{\Delta t}{4\Delta x^2}\left(p_{i+2,j} - 2p_{i,j} + p_{i-2,j}\right)$$

$$+ \frac{\Delta t}{4\Delta y^2}\left(p_{i,j+2} - 2p_{i,j} + p_{i,j-2}\right)$$

Structure of the equation

The equation splits pressure in two colors, as below



The circle points $\bigcirc$ are related to each other and the x points $\times$ points are related to each other. Hence every alternate pressure in i and alternate pressure in j are related! The immediate neighbors are not related to each other. This is called a checker-board pressure splitting or $2\Delta x$ oscillation.

Staggered Velocities

Let us write

$$u^{n+1} = \hat{u} - \frac{1}{\rho}\frac{\partial p}{\partial x}\Delta t\ ; \quad \mathrm{v}^{n+1} = \hat{\mathrm{v}} - \frac{1}{\rho}\frac{\partial p}{\partial y}\Delta t$$

$$\frac{\partial u^{n+1}}{\partial x} + \frac{\partial \mathrm{v}^{n+1}}{\partial y} = \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} - \frac{\Delta t}{\rho}\nabla^2 p$$

$$\nabla^2 p = \frac{\rho}{\Delta t}\nabla \cdot \hat{u}$$

Staggered Velocities

$$\nabla \cdot \hat{u} = \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = \frac{\hat{u}_{i+1,j} - \hat{u}_{i-1,j}}{2\Delta x} + \frac{\hat{\mathrm{v}}_{i,j+1} - \hat{\mathrm{v}}_{i,j-1}}{2\Delta y}$$

$$= \frac{\hat{u}_{i+1,j} - \hat{u}_{i,j}}{2\Delta x} - \frac{\hat{u}_{i,j} + \hat{u}_{i-1,j}}{2\Delta x} + \frac{\hat{v}_{i,j+1} - \hat{v}_{i,j}}{2\Delta y}$$

$$- \frac{\hat{v}_{i,j} - \hat{v}_{i,j-1}}{2\Delta y}$$

$$= \frac{\hat{u}_{i+1/2,j} - \hat{u}_{i-1/2,j}}{\Delta x} + \frac{\hat{v}_{i,j+1/2} - \hat{v}_{i,j-1/2}}{\Delta y}$$

Staggered Velocities

$$\rho \frac{{u_{i,j}}^{n+1} - \hat{u}_{i,j}}{\Delta t} = -\left(\frac{\partial p}{\partial x}\right)^{n+1}$$

Likewise

$$\rho \frac{{\mathrm{v}_{i,j}}^{n+1} - \hat{\mathrm{v}}_{i,j}}{\Delta t} = -\left(\frac{\partial p}{\partial y}\right)^{n+1}$$

Continuity equation (i,j):

$$\frac{{u_{i+1/2,j}}^{n+1} - {u_{i-1/2,j}}^{n+1}}{\Delta x} + \frac{{\mathrm{v}_{i,j+1/2}}^{n+1} - {\mathrm{v}_{i,j-1/2}}^{n+1}}{\Delta y} = 0$$

Staggered Velocities

Now, the important step is to define

${u_{i+1/2,j}}^{n+1}$ **and** ${\mathrm{v}_{i,j+1/2}}^{n+1}$

$$u^{n+1} = \hat{u} - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial x}\right)^{n+1}$$

$$\mathrm{v}^{n+1} = \hat{\mathrm{v}} - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial y}\right)^{n+1}$$

$$u^{n+1}{}_{i+1/2,j} = \hat{u}_{i+1/2,j} - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial x}\right)^{n+1\,(i+1/2,\,j)}$$

$$= \hat{u}_{i+1/2,j} - \frac{\Delta t}{\rho}\left(\frac{{p_{i+1,j}}^{n+1} - {p_{i,j}}^{n+1}}{\Delta x}\right)$$

Continuity Equation

$$u^{n+1}{}_{i-1/2,j} = \hat{u}_{i-1/2,j} - \frac{\Delta t}{\rho}\left(\frac{p_{i,j}{}^{n+1} - p_{i-1,j}{}^{n+1}}{\Delta x}\right)$$

$$\mathrm{v}^{n+1}{}_{i,j+1/2} = \hat{\mathrm{v}}_{i,j+1/2} - \frac{\Delta t}{\rho}\left(\frac{p_{i,j+1}{}^{n+1} - p_{i,j}{}^{n+1}}{\Delta y}\right)$$

Now we can write the continuity equation at (i,j) as

$$\frac{u_{i+1/2,j}{}^{n+1} - u_{i-1/2,j}{}^{n+1}}{\Delta x} + \frac{\mathrm{v}_{i,j+1/2}{}^{n+1} - \mathrm{v}_{i,j-1/2}{}^{n+1}}{\Delta y} =$$

$$\frac{\hat{u}_{i+1/2,j} - \hat{u}_{i-1/2,j}}{\Delta x} + \frac{\hat{\mathrm{v}}_{i,j+1/2} - \hat{\mathrm{v}}_{i,j-1/2}}{\Delta y} \quad -$$

Pressure-Poisson Equation

$$\frac{\Delta t}{\rho \Delta x}\left(\left(\frac{p_{i+1,j}{}^{n+1}-p_{i,j}{}^{n+1}}{\Delta x}\right)-\left(\frac{p_{i,j}{}^{n+1}-p_{i-1,j}{}^{n+1}}{\Delta x}\right)\right)-$$

$$\frac{\Delta t}{\rho \Delta y}\left(\left(\frac{p_{i,j+1}{}^{n+1}-p_{i,j}{}^{n+1}}{\Delta y}\right)-\left(\frac{p_{i,j}{}^{n+1}-p_{i,j-1}{}^{n+1}}{\Delta y}\right)\right)$$

Simplifying:

$$\left(\frac{p_{i+1,j}{}^{n+1}-2p_{i,j}{}^{n+1}+p_{i-1,j}{}^{n+1}}{\Delta x^2}\right)+$$

$$\left(\frac{p_{i,j-1}{}^{n+1}-2p_{i,j}{}^{n+1}+p_{i,j-1}{}^{n+1}}{\Delta y^2}\right)=\frac{\rho}{\Delta t}\left(\frac{\hat{u}_{i+1/2,j}-\hat{u}_{i-1/2,j}}{\Delta x}+\right.$$

Pressure-Poisson Equation

Now we have a well-connected pressure field in which:

$$\hat{u}_{i+1/2,j},\ \hat{u}_{i-1/2,j}, \hat{\mathrm{v}}_{i,j+1/2}, \hat{\mathrm{v}}_{i,j-1/2}$$

satisfy continuity, and

$$\hat{u}_{i+1/2,j} \approx \frac{1}{2}\left(\hat{u}_{i,j} + \hat{u}_{i+1,j}\right)$$

$$\hat{u}_{i-1/2,j} \approx \frac{1}{2}\left(\hat{u}_{i,j} + \hat{u}_{i-1,j}\right)$$

$$\hat{\mathrm{v}}_{i,j+1/2} \approx \frac{1}{2}\left(\hat{\mathrm{v}}_{i,j} + \hat{\mathrm{v}}_{i,j+1}\right)$$

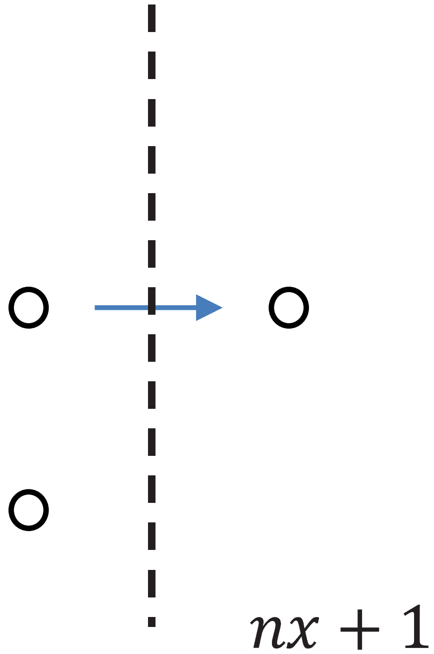
$$\hat{\mathrm{v}}_{i,j-1/2} \approx \frac{1}{2}\left(\hat{\mathrm{v}}_{i,j} + \hat{\mathrm{v}}_{i,j-1}\right)$$

Continuity Equation

$$\boldsymbol{u^{n+1}}_{i,j} = \widehat{\boldsymbol{u}}_{i,j} - \frac{\Delta t}{\rho}\left(\frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}\right)$$

$$\mathbf{v}^{n+1}{}_{i,j} = \hat{\mathbf{v}}_{i,j} - \frac{\Delta t}{\rho}\left(\frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y}\right)$$

The $u^{n+1}{}_{i,j}$ and $\mathbf{v}^{n+1}{}_{i,j}$ DO NOT satisfy discrete continuity. The values at $u^{n+1}{}_{i+\frac{1}{2},j}$ and $v^{n+1}{}_{i,j+\frac{1}{2}}$ satisfy discrete continuity, but not the discrete momentum equations.


$nx + 1$

$$\frac{u_{nx+1/2,j} - \left(\hat{u}_{nx-1/2,j} - \frac{\Delta t}{\rho}\left(\frac{p_{nx,j} - p_{nx-1,j}}{\Delta x}\right)\right)}{\Delta x}$$

$$+ \left(\hat{\mathbf{v}}_{nx,j+\frac{1}{2}} - \hat{\mathbf{v}}_{nx,j-\frac{1}{2}} - \frac{\Delta t}{\rho}\left(\frac{p_{nx,j+1} - p_{nx,j}}{\Delta y}\right)\right)$$

Summary

- **Solve for $\hat{u}$,$\hat{\mathrm{v}}$ at collocated grid node positions**
- **Evaluate $\hat{u}$,$\hat{\mathrm{v}}$ at (i+1/2,j), (i,j+1/2) positions by averaging**
- **Evaluate discrete continuity errors in $\hat{u}$,$\hat{\mathrm{v}}$ at (i,j).**
- **Solve pressure Poisson equation at grid nodes (i,j)**
- **Correct grid node velocities $u^{n+1}{}_{i,j}$, $\mathrm{v}^{n+1}{}_{i,j}$ using $\hat{u}_{i,j}$,$\hat{\mathrm{v}}_{i,j}$ and $p_{i-1,j}$, $p_{i+1,j}$, $p_{i,j+1}$, $p_{i,j-1}$**
- **March to next time step**
- **Momentum equations are explicit**
- **Pressure equations solved by SOR or other iterative solvers.**

Fractional Step Method with Implicit Diffusion

Momentum Equations with Implicit Diffusion

$$\rho \frac{\hat{u} - u^n}{\Delta t} = -(conv)_x^{\,n} + \widehat{(Diff)}_x$$

$$\rho \frac{\hat{\mathrm{v}} - v^n}{\Delta t} = -(conv)_y^{\,n} + \widehat{(Diff)}_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{\mathrm{v}}}{\partial x}\right)$$

Φ and p are not the same, but related.

Momentum Equations

$$\rho \frac{u^{n+1}-u^{n}}{\Delta t} = -\frac{\partial p}{\partial x} + \mu \nabla^{2} u^{n} - (conv)_{x}^{\;n}$$

$$\rho \frac{v^{n+1}-v^{n}}{\Delta t} = -\frac{\partial p}{\partial y} + \mu \nabla^{2} v^{n} - (conv)_{y}^{\;n}$$

$$\rho \frac{\hat{u}-u^{n}}{\Delta t} = -(conv)_{x}^{\;n} + \mu \nabla^{2} \hat{u}$$

$$u^{n+1} = \hat{u} - \frac{\partial \Phi}{\partial x} \cdot \frac{\Delta t}{\rho}$$

$$\mu \nabla^{2} u^{n+1} = \mu \nabla^{2} \hat{u} - \mu \frac{\Delta t}{\rho} \nabla^{2} \left(\frac{\partial \Phi}{\partial x} \right)$$

$$\mu \nabla^{2} v^{n+1} = \mu \nabla^{2} \hat{v} - \mu \frac{\Delta t}{\rho} \nabla^{2} \left(\frac{\partial \Phi}{\partial y} \right)$$

Momentum Equations

$$\rho \frac{u^{n+1}-u^{n}}{\Delta t} = -\frac{\partial \Phi}{\partial x} + \mu \nabla^2 \hat{u} - (conv)_x^{\ n}$$

$$\rho \frac{v^{n+1}-v^{n}}{\Delta t} = -\frac{\partial \Phi}{\partial y} + \mu \nabla^2 \hat{v} - (conv)_y^{\ n}$$

Subtracting the u equations from each other

$$\mu \nabla^2 (\hat{u} - u^n) = \frac{\partial}{\partial x}(-p + \Phi)$$

$$\mu \nabla^2 (\hat{v} - v^n) = \frac{\partial}{\partial y}(-p + \Phi)$$

It can be shown that $p = \Phi - \mu \frac{\Delta t}{\rho} \nabla^2 \Phi$

Relative Nature of Φ (and p).

The pressure Poisson equation is ill-conditioned in the sense that there is <u>one less</u> equation. The sum of all local continuity equations is the global continuity equation. Hence if the Φ (or p) equations is solved <u>directly</u>, it will come up with an error. Only iterative solvers can be used. Another option is to add another equation which fixes the pressure level at any arbitrary location.

$$p\left(i_{ref}, j_{ref}\right) = p_{ref}$$

Adams-Bashforth Differencing

$$\rho \frac{\hat{u}-u^n}{\Delta t} = -\frac{3}{2}(conv)\,{}_x{}^{n} + \frac{1}{2}(conv)\,{}_x{}^{n-1} + \frac{1}{2}\left(\mu\nabla^2\hat{u} + \mu\nabla^2 u^n\right)$$

$$\rho \frac{\hat{\mathrm{v}}-u^n}{\Delta t} = -\frac{3}{2}(conv)\,{}_y{}^{n} + \frac{1}{2}(conv)\,{}_y{}^{n-1} + \frac{1}{2}\left(\mu\nabla^2\hat{\mathrm{v}} + \mu\nabla^2 \mathrm{v}^n\right)$$

$$\left.\begin{aligned} u^{n+1} &= \hat{u} - \frac{\partial \Phi}{\partial x}\cdot\frac{\Delta t}{\rho} \\ v^{n+1} &= \hat{v} - \frac{\partial \Phi}{\partial y}\cdot\frac{\Delta t}{\rho} \end{aligned}\right\}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\rho}{\Delta t}\left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{\mathrm{v}}}{\partial y}\right)$$

Implicit Convection Schemes

If time accuracy is not of much concern, we should be able to take larger time steps either to reach steady state quickly, or to study longer transients. In such cases, explicit schemes for convection <u>may</u> not be optimal. Implicit diffusion and implicit convection schemes may take less time overall, although work per time step can be greater.

The SIMPLE Algorithm

SIMPLE

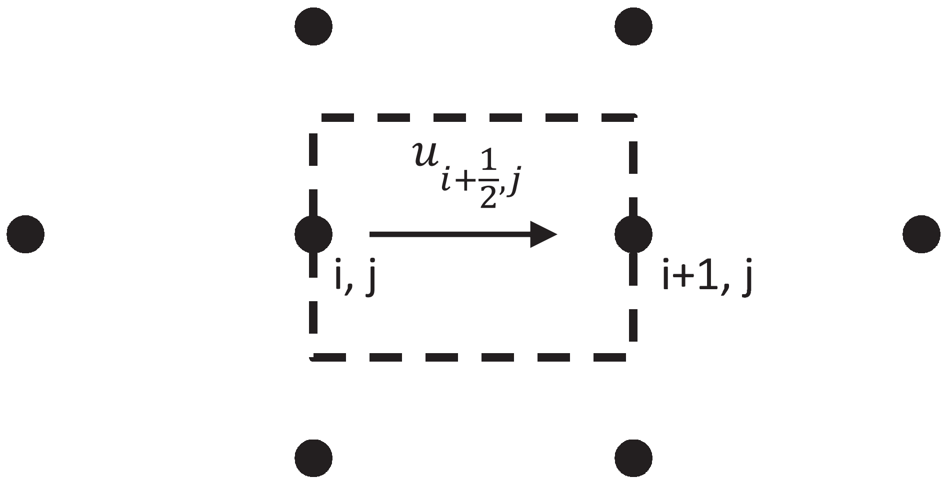
SIMPLE stands for Semi-Implicit-Method for Pressure Linked Equations. It is a finite volume method that uses a staggered grid.

Iterative vs. time marching

Primary emphasis on steady-state flow fields compared to time evolution. Can be used for long term transients.

SIMPLE is a guess and correct algorithm to satisfy the momentum and continuity equations.

Staggered Mesh



Define a control volume around (i+1/2, j). The control volume spans two pressure nodes (i, j) and (i+1, j). In the x-direction and (i, j-1/2) to

(i, j+1/2) in the y-direction. The u velocity is centered between two pressure nodes.

Continuity Equation

The continuity equation uses C_x, C_y at (i+1/2, j), (i-1/2, j) etc.

$$(C_x)_{i+\frac{1}{2},j} - (C_x)_{i-\frac{1}{2},j} + (C_y)_{i,j+\frac{1}{2}} - (C_y)_{i,j-\frac{1}{2}} = 0$$

$$(C_x)_{i+\frac{1}{2},j} = \int_S (\rho u \Delta y)_{i+\frac{1}{2},j}$$

$$(C_x)_{i-\frac{1}{2},j} = \int_S (\rho u \Delta y)_{i-\frac{1}{2},j}$$

$$(C_y)_{i,j+\frac{1}{2}} = \int_S (\rho \mathrm{v} \Delta x)_{i,j+\frac{1}{2}}$$

$$(C_y)_{i,j-\frac{1}{2}} = \int_S (\rho v \Delta x)_{i,j-\frac{1}{2}}$$

Advection terms

The x-momentum convection terms can be written as

$$\iint \frac{\partial}{\partial x}(\rho u u)dxdy + \iint \frac{\partial}{\partial y}(\rho u \mathrm{v})dxdy$$

$$\approx (C_x)_{i+1,j}u_{i+1,j} - (C_x)_{i,j}u_{i,j}$$

$$+\left(C_y\right)_{i+\frac{1}{2},j+\frac{1}{2}}u_{i+\frac{1}{2},j+\frac{1}{2}} - \left(C_y\right)_{i+\frac{1}{2},j-\frac{1}{2}}u_{i+\frac{1}{2},j-\frac{1}{2}}$$

For the mass fluxes, we must use normal averaging

Linear Interpolation

What are the values of $u_{i,j}$, $u_{i+1,j}$, $u_{i+\frac{1}{2},j+\frac{1}{2}}$, $u_{i+\frac{1}{2},j-\frac{1}{2}}$?

If we use central differencing, or linear averaging, then

$$u_{i,j} = \frac{1}{2}\left(u_{i+1/2,j} + u_{i-1/2,j}\right)$$

$$u_{i+1,j} = \frac{1}{2}\left(u_{i+3/2,j} + u_{i+1/2,j}\right)$$

$$u_{i+1/2,j+1/2} = \frac{1}{2}\left(u_{i+1/2,j} + u_{i+1/2,j+1}\right)$$

$$u_{i+1/2,j-1/2} = \frac{1}{2}\left(u_{i+1/2,j} + u_{i+1/2,j-1}\right)$$

Upwinding

$If C_{x+}\ is > 0 \qquad u_{i+1,j} = u_{i+1/2,j}$

$If C_{x+}\ is < 0 \qquad u_{i+1,j} = u_{i+3/2,j}$

Likewise

$If C_{x-}\ is > 0 \qquad u_{i,j} = u_{i-1/2,j}$

$If C_{x-}\ is < 0 \qquad u_{i,j} = u_{i+1/2,j}$

Similarly in the y direction

Diffusion Terms

Diffusion terms central differenced as before

$$\iint_V \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) dxdy = \mu \frac{\partial u}{\partial x}\Delta y\bigg|_{x+} - \mu \frac{\partial u}{\partial x}\Delta y\bigg|_{x-}$$

$$= \mu_{i+1,j}\frac{\Delta y}{\Delta x}\left(u_{i+\frac{3}{2},j} - u_{i+\frac{1}{2},j}\right) - \mu_{i,j}\frac{\Delta y}{\Delta x}\left(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\right)$$

$$\iint_V \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dxdy = \mu \frac{\partial u}{\partial y}\Delta x\bigg|_{y+} - \mu \frac{\partial u}{\partial y}\Delta x\bigg|_{y-}$$

$$= \mu_{y+}\frac{\Delta x}{\Delta y}\left(u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j}\right) - \mu_{y-}\frac{\Delta x}{\Delta y}\left(u_{i+\frac{1}{2},j} - u_{i+\frac{1}{2},j-1}\right)$$

Complete Equation

Assemble the complete equation as

$$
\begin{aligned}
A_p{}^u\; u_{i+\frac{1}{2},j} = {} & A_W{}^u\; u_{i-\frac{1}{2},j} + A_E{}^u\, u_{i+3/2,j} \\
& + A_S{}^u\; u_{i+\frac{1}{2},j-1} + A_N{}^u\; u_{i+\frac{1}{2},j+1} \\
& + \left(p_{i,j} - p_{i+1,j}\right)\Delta y
\end{aligned}
$$

Where the coefficients can be written as (for central differencing)

$$
\begin{aligned}
A_W{}^u &= \left(\frac{\mu \Delta y}{\Delta x}\right)_{i,j} + \rho_{i,j}\frac{\Delta y}{2} u_{i,j} \\
&= \frac{1}{2}(C_x)_{i,j} + \left(\frac{\mu \Delta y}{\Delta x}\right)_{i,j}
\end{aligned}
$$

Coefficients

$$A_E{}^u = \left(\frac{\mu\Delta y}{\Delta x}\right)_{i+1,j} - \rho_{i+1,j}\frac{\Delta y}{2}u_{i+1,j}$$

$$= -\frac{1}{2}(C_x)_{i+1,j} + \left(\frac{\mu\Delta y}{\Delta x}\right)_{i+1,j}$$

$$A_S{}^u = \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}} - \rho_{i+\frac{1}{2},j-\frac{1}{2}} \cdot \Delta x \mathbf{v}_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$= \frac{1}{2}(C_y)_{i+\frac{1}{2},j-\frac{1}{2}} + \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$A_N{}^u = -\frac{1}{2}(C_y)_{i+\frac{1}{2},j+\frac{1}{2}} + \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j+\frac{1}{2}}$$

Coefficients with Upwinding

$$if\ C_x > 0 \qquad A_W{}^u = (C_x)_{i,j} + \left(\frac{\mu\Delta y}{\Delta x}\right)_{i,j}$$

$$if\ C_x < 0 \qquad A_W{}^u = \left(\frac{\mu\Delta y}{\Delta x}\right)_{i,j}$$

$$if (C_x)_{i+1,j} > 0 \qquad A_E{}^u = \left(\frac{\mu\Delta y}{\Delta x}\right)_{i+1,j}$$

$$if (C_x)_{i+1,j} < 0 \qquad A_E{}^u = (-C_x)_{i+1,j} + \left(\frac{\mu\Delta y}{\Delta x}\right)_{i+1,j}$$

$$if\left(C_y\right)_{i+\frac{1}{2},j-\frac{1}{2}} > 0 \quad A_S{}^u = \left(C_y\right)_{i+\frac{1}{2},j-\frac{1}{2}} + \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}}$$

$$if\left(C_y\right)_{i+\frac{1}{2},j-\frac{1}{2}} < 0 \qquad A_S{}^u = \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}}$$

Central coefficient

What about A_p? : A_p can be assembled in a similar way by collecting coefficients.

$$A_p = \left(\frac{\mu\Delta y}{\Delta x}\right)_{i+1,j} + \left(\frac{\mu\Delta y}{\Delta x}\right)_{i,j} + \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j+\frac{1}{2}} + \left(\frac{\mu\Delta x}{\Delta y}\right)_{i+\frac{1}{2},j-\frac{1}{2}} - \frac{1}{2}(C_x)_{i,j} + \frac{1}{2}(C_x)_{i+1,j} - \frac{1}{2}(C_y)_{i+\frac{1}{2},j-\frac{1}{2}} + \frac{1}{2}(C_y)_{i+\frac{1}{2},j+\frac{1}{2}}$$

Central coefficient

Let us make A_p diagonally dominant, by implicitly invoking continuity to the momentum equation. Then we can subtract this from A_p and show that

$$A_p{}^u = A_W{}^u + A_E{}^u + A_S{}^u + A_N{}^u$$

Continuity equation

$$\iint_V \left\{ \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) \right\} dxdy = 0$$

$$(\rho u \Delta y)_{i+\frac{1}{2},j} - (\rho u \Delta y)_{i-\frac{1}{2},j} + (\rho \mathrm{v} \Delta x)_{i,j+\frac{1}{2}} - (\rho v \Delta x)_{i,j-\frac{1}{2}} = 0$$

$$(C_x)_{i+\frac{1}{2},j} - (C_x)_{i-\frac{1}{2},j} + \left(C_y\right)_{i,j+\frac{1}{2}} - \left(C_y\right)_{i,j-\frac{1}{2}} = 0$$

Truncated Momentum Equations

$$A_p{}^u\ u_{i+\frac{1}{2},j} = A_W{}^u\ u_{i-\frac{1}{2},j} + A_E{}^u\ u_{i+\frac{3}{2},j} + A_S{}^u\ u_{i+\frac{1}{2},j-1} + A_N{}^u\ u_{i+\frac{1}{2},j+1} + \left(p_{i,j} - p_{i+1,j}\right)\Delta y + \Delta x \Delta y \bar{S}$$

$$A_p{}^u u'_{i+\frac{1}{2},j} = A_W{}^u u'_{i-\frac{1}{2},j} + A_E{}^u u'_{i+\frac{1}{2},j} + A_S{}^u u'_{i+\frac{1}{2},j-1} + A_N{}^u u'_{i+\frac{1}{2},j+1} + \left(p'_{i,j} - p'_{i+1,j}\right)\Delta y + \Delta x \Delta y \bar{S}'$$

Truncated Momentum Equations

Let us truncate this and write this as

$$A_p{}^u u'_{i+\frac{1}{2},j} \approx \left(p'_{i,j} - p'_{i+1,j}\right)\Delta y$$

Likewise

$$A_p{}^u u'_{i-\frac{1}{2},j} \approx \left(p'_{i-1,j} - p'_{i,j}\right)\Delta y$$

$$A_p{}^{\mathrm{v}} \mathrm{v}'_{i,j+\frac{1}{2}} \approx \left(p'_{i,j-1} - p'_{i,j}\right)\Delta x$$

$$A_p{}^v v'_{i,j-\frac{1}{2}} \approx \left(p'_{i,j} - p'_{i,j+1}\right)\Delta x$$

Continuity equation

Write in terms of * and ' values

$$\boldsymbol{\rho}_{i+\frac{1}{2},j}(\boldsymbol{u}^{*}+\boldsymbol{u}')_{i+\frac{1}{2},j}\Delta \boldsymbol{y}-\boldsymbol{\rho}_{i-\frac{1}{2},j}(\boldsymbol{u}^{*}+\boldsymbol{u}')_{i-\frac{1}{2},j}\Delta \boldsymbol{y}+\boldsymbol{\rho}_{i,j+\frac{1}{2}}(\mathbf{v}^{*}+\mathbf{v}')_{i,j+\frac{1}{2}}\Delta \boldsymbol{x}-\boldsymbol{\rho}_{i,j-\frac{1}{2}}(\boldsymbol{v}^{*}+\boldsymbol{v}')_{i,j-\frac{1}{2}}\Delta \boldsymbol{x}=\mathbf{0}$$

Substitute the expressions for $\boldsymbol{u}'$, $\mathbf{v}'$ as pressure corrections.

Pressure correction equation

$$\rho_{i+\frac{1}{2},j}\Delta y\left(\frac{\Delta y}{\left(A_p{}^{u}\right)_{i+\frac{1}{2},j}}\right)\left(p'_{i,j}-p'_{i+1,j}\right)-$$
$$\rho_{i-\frac{1}{2},j}\Delta y\left(\frac{\Delta y}{\left(A_p{}^{u}\right)_{i-\frac{1}{2},j}}\right)\left(p'_{i-1,j}-p'_{i,j}\right)+$$
$$\rho_{i,j+\frac{1}{2}}\Delta x\left(\frac{\Delta x}{\left(A_p{}^{\mathrm{v}}\right)_{i,j+\frac{1}{2}}}\right)\left(p'_{i,j}-p'_{i,j+1}\right)-$$
$$\rho_{i,j-\frac{1}{2}}\Delta x\left(\frac{\Delta x}{\left(A_p{}^{v}\right)_{i,j-\frac{1}{2}}}\right)\left(p'_{i,j-1}-p'_{i,j}\right)=$$
$$\rho_{i-\frac{1}{2},j}u^*{}_{i-\frac{1}{2},j}\Delta y-\rho_{i+\frac{1}{2},j}u^*{}_{i+\frac{1}{2},j}\Delta y+$$
$$\rho_{i,j-\frac{1}{2}}v^*{}_{i,j-\frac{1}{2}}\Delta x-\rho_{i,j+\frac{1}{2}}v^*{}_{i,j+\frac{1}{2}}\Delta x=S_m$$

Steps in SIMPLE

1. Guess u, v, p
2. Form time loop
3. Solve for u*, v*
4. Solve for p’
5. Correct $u = u^* + u'$; $v = v^* + v'$; $p = p^* + p'$;
6. Repeat 3,4,5 until convergence
7. Increment time step, solve 3-6
8. Complete time loop

Multigrid Methods

What are multigrid methods

- Multigrid methods are efficient solvers for such sets of equations and provide rapid convergence.
- The multigrid technique is based on the strategy that the low frequency errors of any grid can be solved on a coarser grid at a faster rate. The finer grid solution can then be corrected for the low frequency errors.
- A large amount of literature exists on multigrid solution of model elliptic partial-differential equations.

Single Grid Convergence

- Each frequency of error converges at a different rate. High frequency errors converge the fastest. Low frequency errors converge the slowest. Convergence is held up by the long wavelength, low frequency components of the error.
- Each iterative procedure has different rates of convergence for high and low frequencies. Point operators are the cheapest in work count, but convergence is slow for low frequencies.

Introduction to the multigrid concept

- The objective is to preserve the high frequency rate of convergence on all grids. Then, if we have a good solver for the coarse grid, we should be able to get a fast convergence on finer grids.
- In multigrid method, we have several grids. Each grid differs from its coarser/finer grid by a factor of two or more/less number of cells in each direction.
- The objective is to use all these grids in a continuous cycling procedure such that the errors in the solution are resolved efficiently.

Multigrid technique

The multigrid technique consists of

- Solving for the high frequency errors
- Calculating the low frequency residuals
- Transferring the residuals to coarser grids
- Solving for the corrections on coarser grid
- Transferring corrections to finer grid

Components of multigrid technique

The main components of a multigrid procedure are:

- Relaxation (iterative solution procedure)
- Restriction (transferring errors to coarser grids)
- Prolongation (correcting fine grid solution using coarse grid corrections)
- Cycling (visiting the grids)

Multigrid theory

$$L\Phi = F$$

Its discretization is given by

$$L^h\Phi^h = F^{\ h}$$

$$R^{\ h} = F^{\ h} - L^h\Phi^h$$

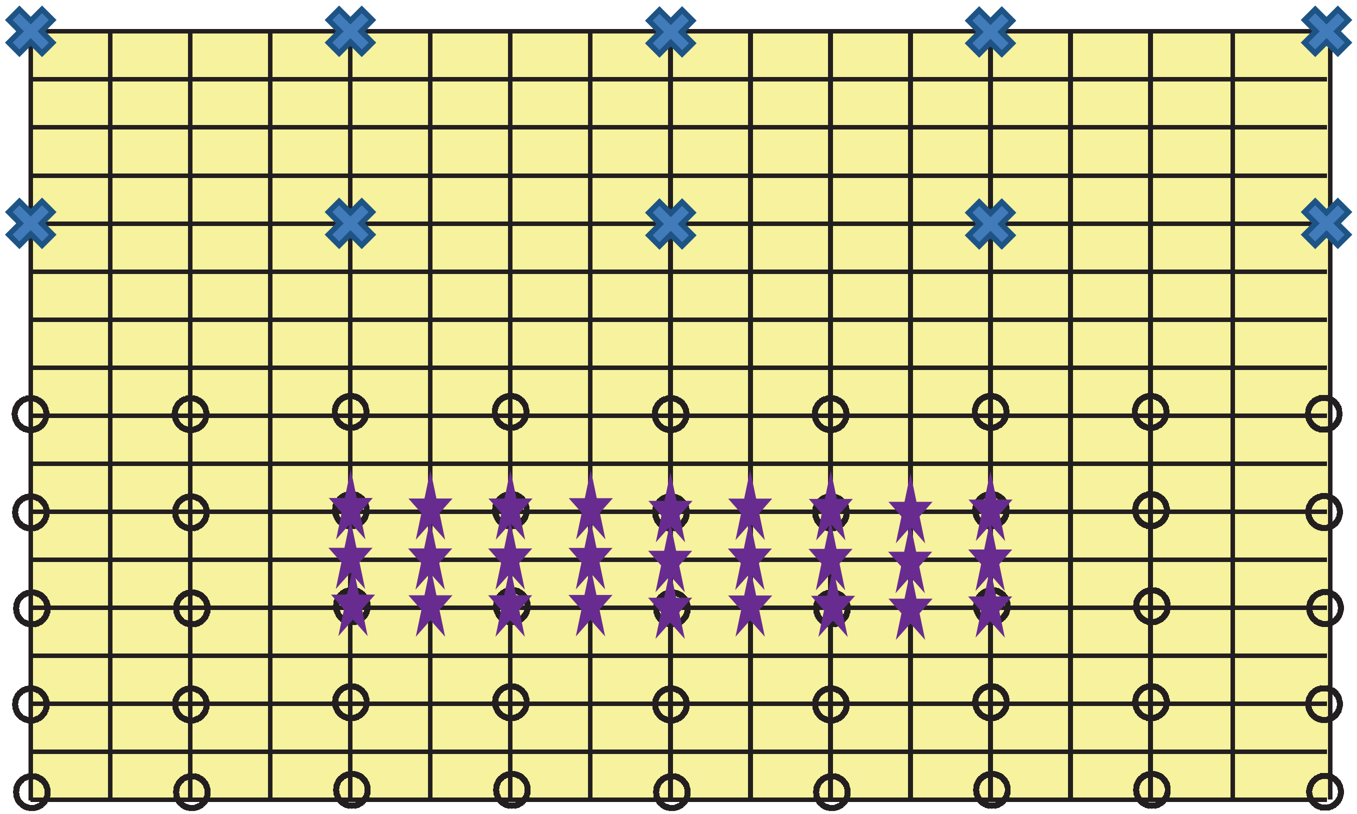
$$R^{\ h-1} = I_{\mathrm{h}}^{\ h-1} \ \ R^{\ h}$$

$$L^{h-1}\delta\phi^{h-1} = R^{h-1} = I_h^{\ h-1}R^h$$

L^{h-1} is transport operator on grid (*h*-1)

$$\phi^h = \phi^h + I_{h\ -\ 1}^{\ h} \ \ \delta\phi^{h-1}$$

2D Finite Difference Grids



grid 1 o grid 2 grid 3

Restriction (Weighted Injection)

$$
\begin{aligned}
&R_c(i_c, j_c) \\
&= \frac{1}{2} R_f(i_f, j_f) \quad + \frac{1}{8} R_f(i_{f-1}, j_f) + \frac{1}{8} R_f(i_{f+1}, j_f) \\
&+ \frac{1}{8} R_f(i_f, j_{f-1}) + \frac{1}{8} R_f(i_f, j_{f+1})
\end{aligned}
$$

$$
\begin{aligned}
i_f &= 2 \cdot i_c - 1 \\
j_f &= 2 \cdot j_c - 1
\end{aligned}
$$

2D Prolongation

For a given coarse grid node, 4 values are updated

$$\phi_f(i_f, j_f) = \phi_f(i_f, j_f) + \delta\phi_c(i_c, j_c)$$

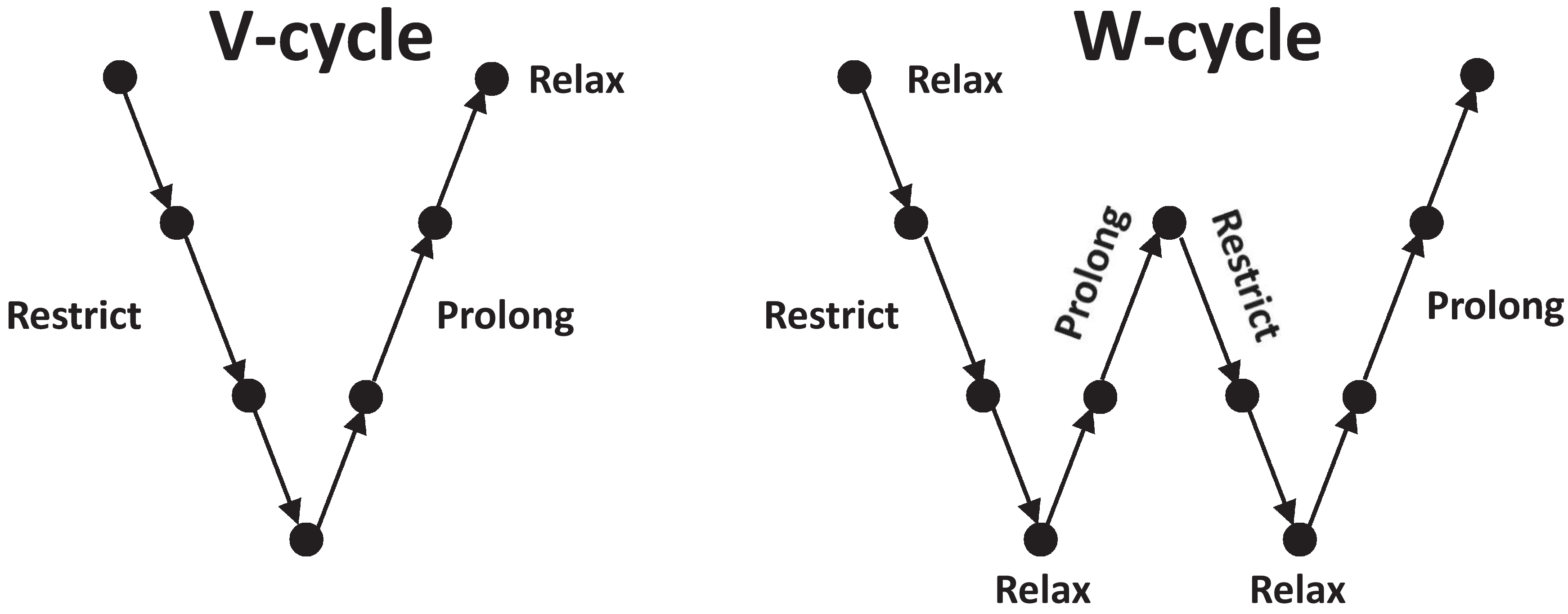
$$\phi_f(i_{f+1}, j_f) = \phi_f(i_{f+1}, j_f) + \frac{1}{2}\delta\phi_c(i_c, j_c) + \frac{1}{2}\delta\phi_c(i_{c+1}, j_c)$$

$$\phi_f(i_f, j_{f+1}) = \phi_f(i_f, j_{f+1}) + \frac{1}{2}\delta\phi_c(i_c, j_c) + \frac{1}{2}\delta\phi_c(i_c, j_{c+1})$$

$$\phi_f(i_{f+1}, j_{f+1}) = \phi_f(i_{f+1}, j_{f+1}) + \frac{1}{4}\delta\phi_c(i_c, j_c) + \frac{1}{4}\delta\phi_c(i_{c+1}, j_{c+1}) + \frac{1}{4}\delta\phi_c(i_c, j_{c+1}) + \frac{1}{4}\delta\phi_c(i_{c+1}, j_{c+1})$$

V- and W-cycles

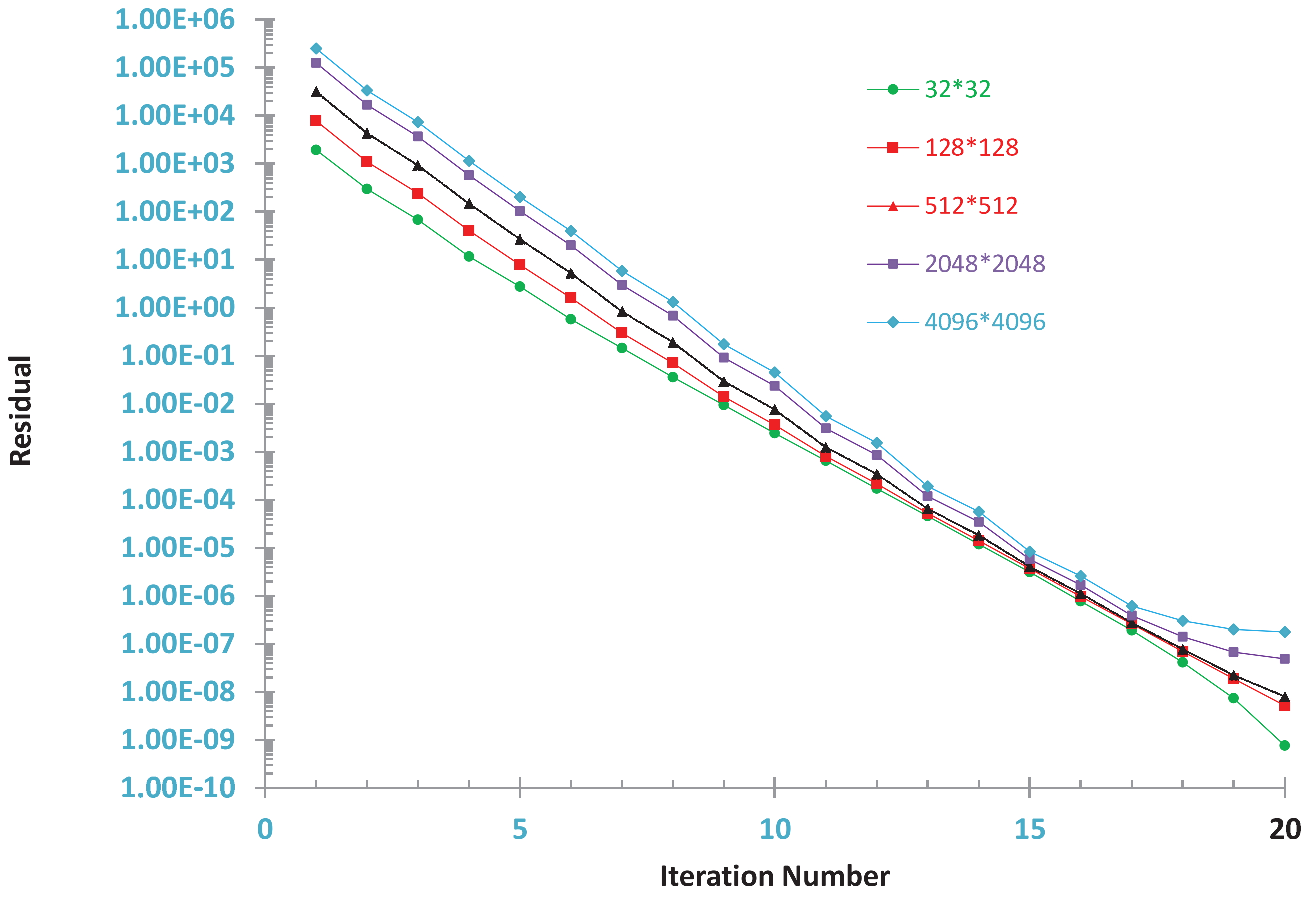
This describes the pattern of visiting the grids. We consider two such strategies.



Benefits of Multigrid Cycling

- All frequencies of error are resolved efficiently.
- The solution efficiency depends only on the coarsest grid solver and is independent of the number of the grid levels.
- The work count is predictable if the coarse grid efficiency is known a priori.

Multigrid convergence of heat conduction equation with Dirichlet conditions



Fluid Flow Equations

- Nonlinear
- Coupled set
- Pressure velocity coupling
- Many scalar equations in addition to momentum and continuity
- Flows can be high or low speed, laminar or turbulent, reacting, complex geometry, etc.

Nonlinear Multigrid

- Multigrid methods for nonlinear equations such as the fluid flow equations can be developed in the same way as the linear multigrid except that the L operator is a function of the solution vector (q). We can develop a nonlinear multigrid by solving for corrections and evaluating the L operator using the solution vector at the top of the V-cycle, i.e.

Nonlinear Multigrid

$$L^h q^h \;=\; f^h$$

$$L^{h-1} \delta q^{h-1} \;=\; I_h^{h-1} R^h = I_h^{h-1}(f^h - L^h q^h)$$

$$L^{h-1} \;=\; L\,(I_h^{h-1} q^h\,)$$

Once δq^{h-1} is computed, it is prolongated to grid h by the prolongation operator I_{h-1}^{h}

$$q^h \;=\; q^h + \, I_{h-1}^{h}\; \delta q^{h-1}$$

FAS Scheme

$$L^{h-1}\delta q^{h-1} \;=\; I_h^{h-1}R^h = I_h^{h-1}(f^h - L^h q^h)$$

Add $L^{h-1}q^{h-1}$ to both sides

$$L^{h-1}(q^{h-1} + \;\delta q^{h-1}) \;=\; I_h^{h-1}R^h \; + L^{h-1}q^{h-1}$$

$$= I_h^{h-1}R^h \; + L^{h-1}q^{h-1} - f^{h-1} + f^{h-1}$$

$$= \; I_h^{h-1}R^h \; - R^{h-1} + f^{h-1}$$

$$= (f^{h-1})_{\text{modified}}$$

FAS Scheme

- We can now solve for the solution vector q^{h-1} with the same discretization sequence as for the general equation
- Solving for q^{h-1} means that the same code structure can be used for every grid. Also, the nonlinearities can be updated for faster convergence. R^{h-1} is the residual in the interpolated solution. It is calculated from $I_h^{h-1} q^h$ only once.
- $q^h = q^h + I_{h-1}^{h} \delta q^{h-1} = I_{h-1}^{h} (q^{h-1} - I_h^{h-1} q^h$

Multiple Levels

If there are more than two levels, the same logic should be used on all coarse grids. However, when grids (h-2), (h-3) etc. are considered, the residuals that are restricted from (h-1) must be calculated using
(

Fluid Flow Equations

- Fluid Flow equations are a strongly coupled set of equations. For good multigrid efficiency, not only the transport operator, but also the pressure-velocity coupling must be resolved across all wavelengths. Simply solving one equation at a time efficiently is NOT sufficient to remove the low frequencies in the coupling. Basically, the complete set of equations must be relaxed in a coupled manner and with the multigrid cycling

Finite Volume Equations

- $A_P u_P = \sum_{nb} A_{nb}\, u_{nb} + D^u\, (\Delta p)_x + S^u$
- $A_P v_P = \sum_{nb} A_{nb}\, v_{nb} + D^v\, (\Delta p)_y + S^v$

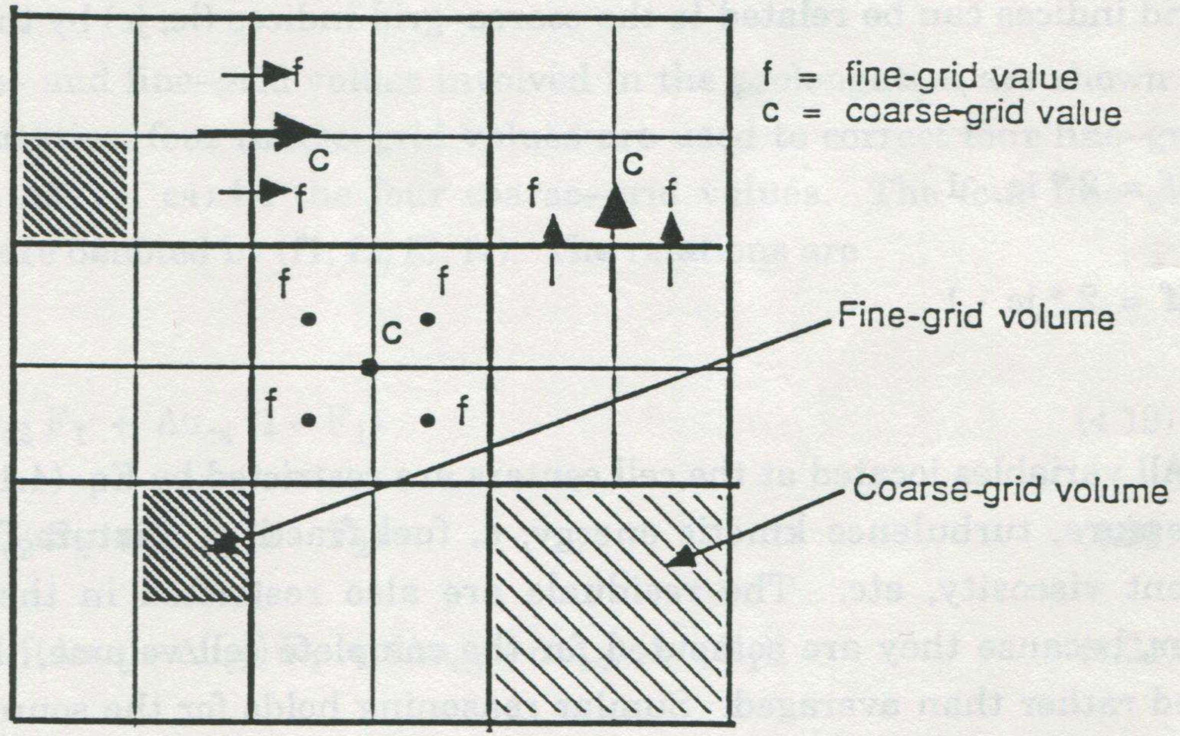
A pressure or pressure correction equation is solved to calculate the pressure field, given by

$$A_P p_P = \sum_{nb} A_{nb}\, p_{nb} + S^p$$

The pressure correction equation can be written as

$$A_P p'_P = \sum_{nb} A_{nb}\, p'_{nb} + S^m$$

Restriction Process



Implementation of a Multigrid SIMPLE

- The SIMPLE algorithm solves the two- (and three-) dimensional fluid flow equations using an implicit relaxation procedure.
- In contrast with a time-marching procedure such as the fractional step method, the SIMPLE algorithm solves directly for the steady state (or quasi-steady) flow fields by iteratively updating the velocities and pressure fields.

Multigrid Cycle

Begin: *ig* = finest grid number = 1

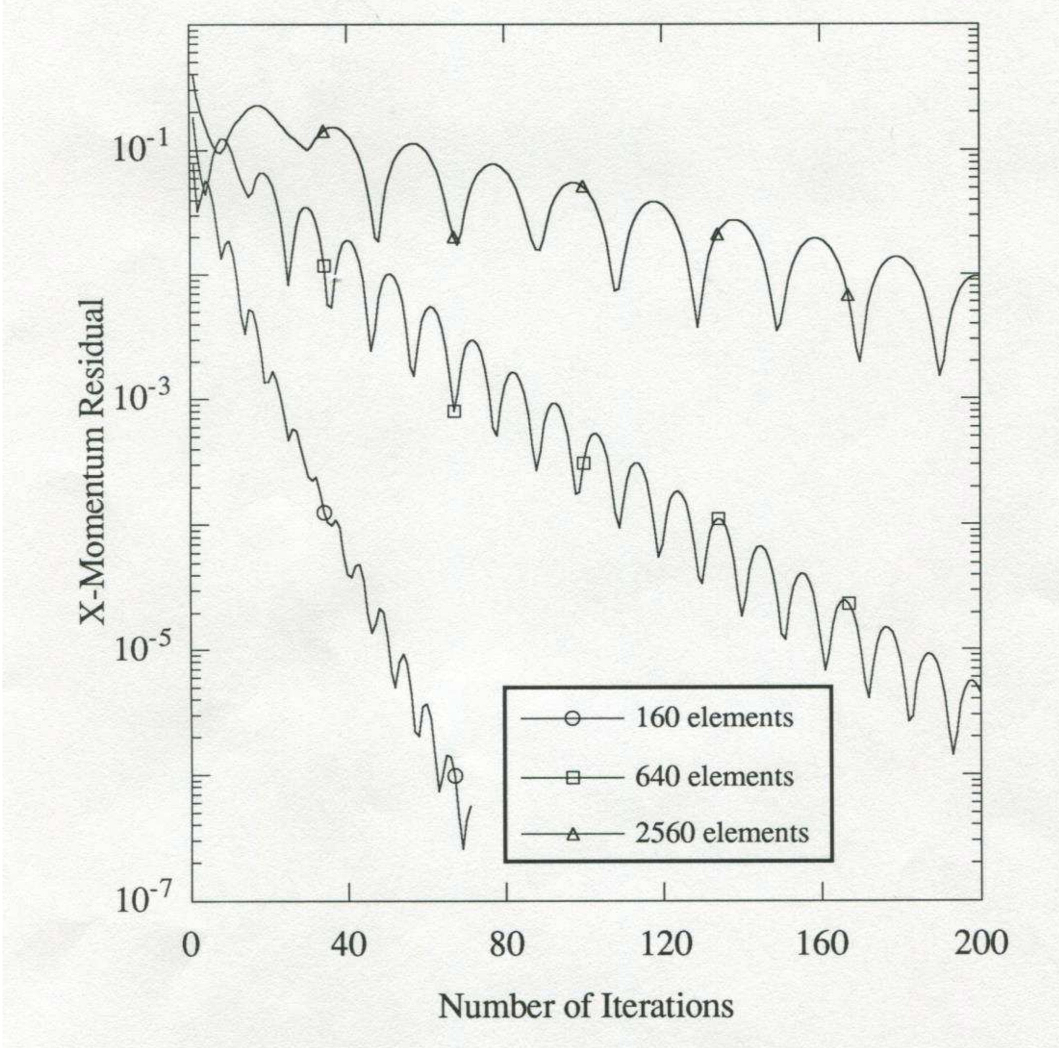
a) Perform relaxations on grid *ig*
b) Calculate residuals on grid *ig*
c) Restrict solution computed on grid *ig* to grid *ig* + 1
d) Restrict residual to grid *ig* + 1
e) Calculate residual on grid *ig* + 1
f) Calculate right hand side of the coarse grid equation
g) Repeat steps (a) to (f) until the coarsest grid is reached

The up leg consists of

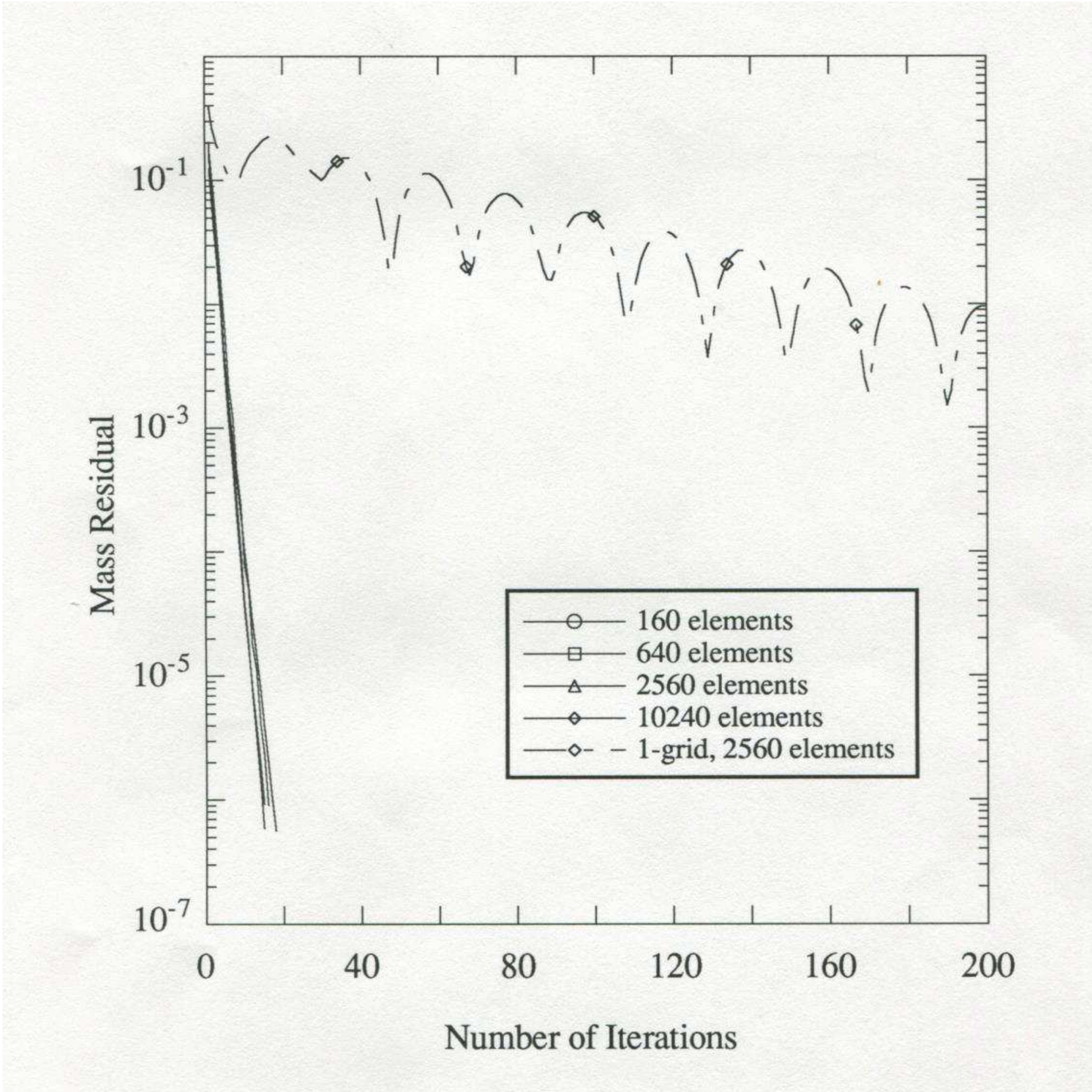
Begin: *ig* = coarsest grid = *ngrid*

a) Perform relaxations on grid *ig*
b) Calculate changes from restricted finer grid *ig* + 1 solution
c) Prolongate changes from the coarser grid to the existing solution on the finer grid
d) Decrement grid number *ig* = *ig* - 1 and repeat steps (a) to (c) until *ig* = 1

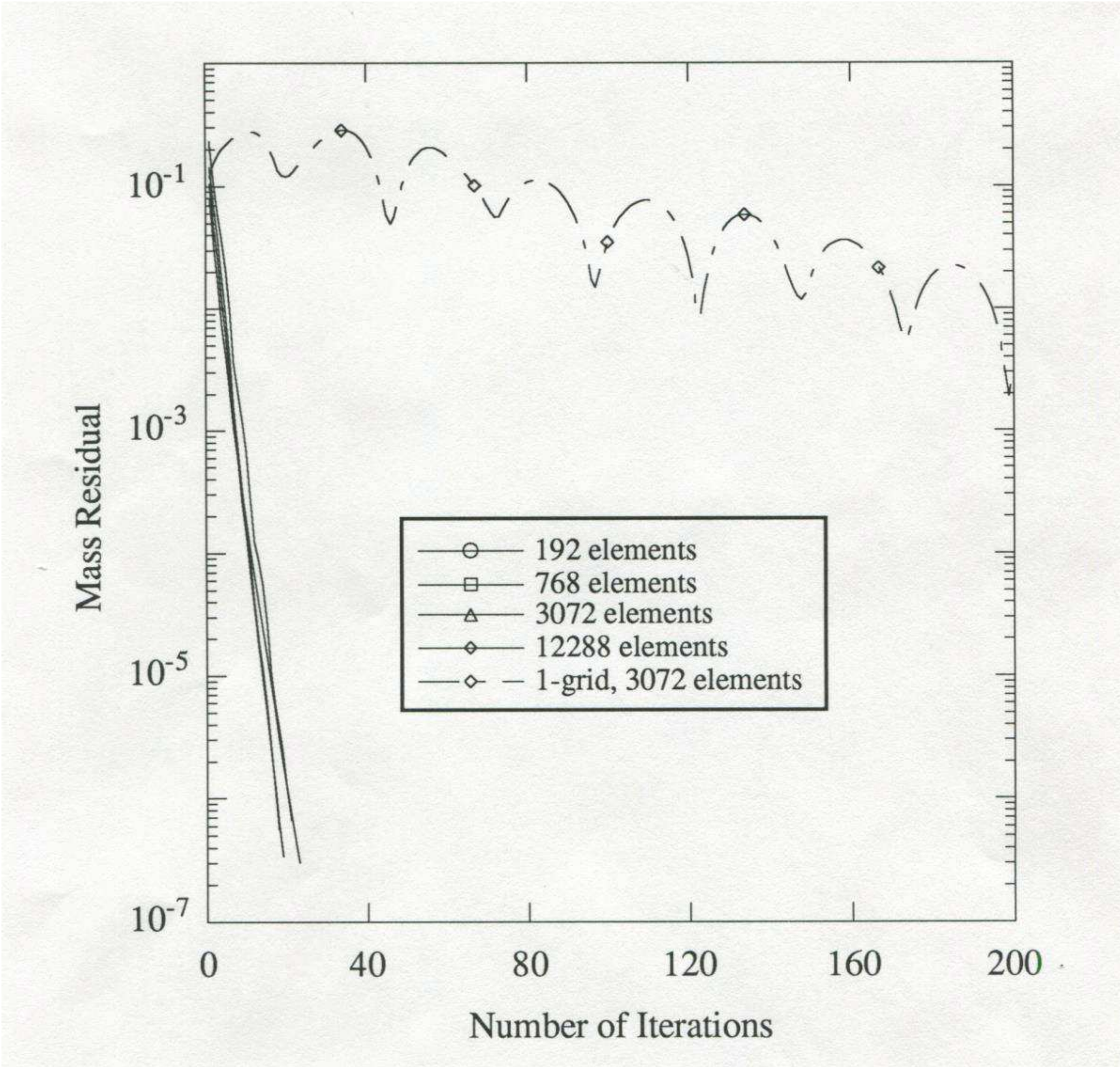
Single Grid Rate of Convergence
Square Cavity



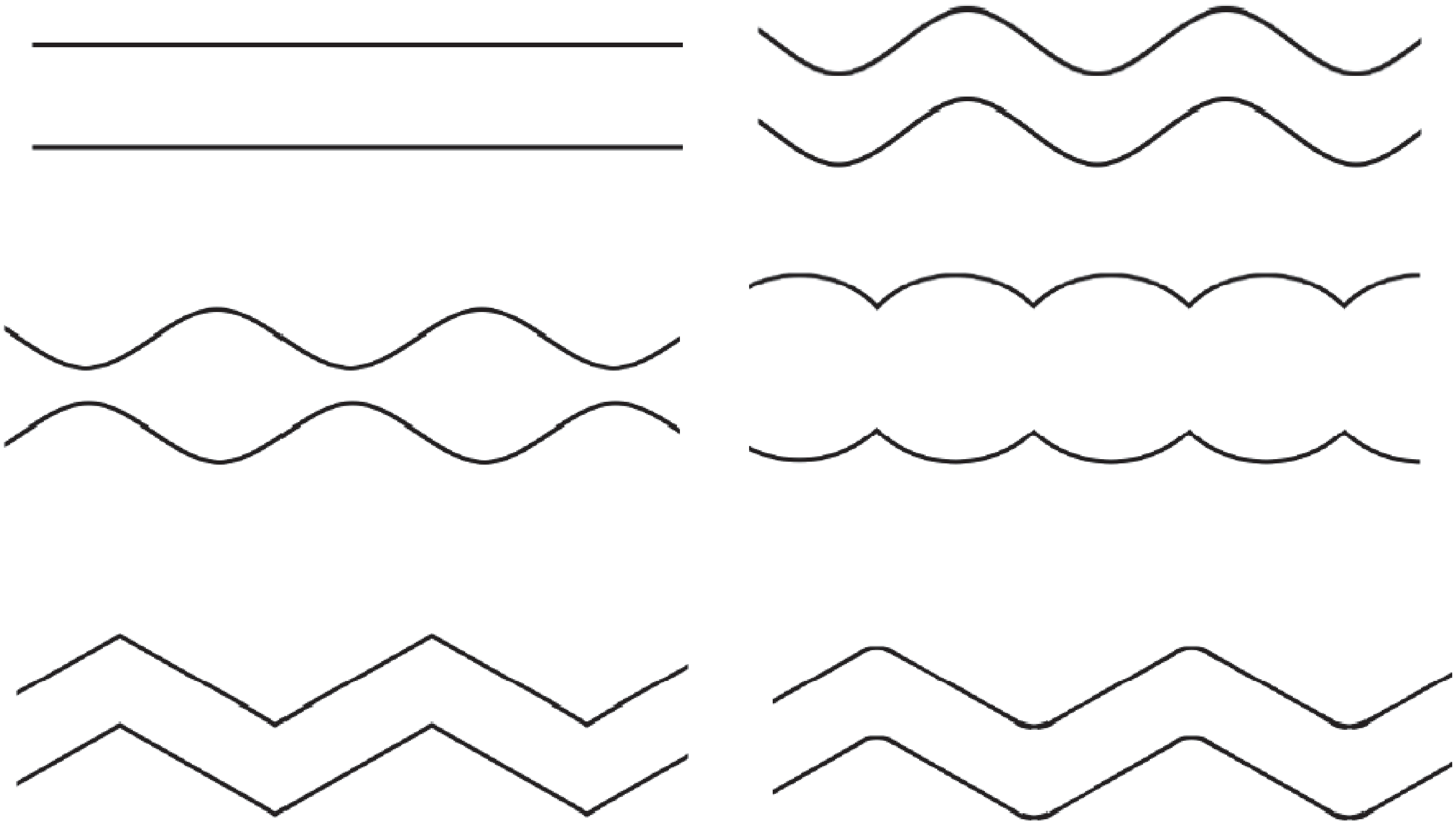
Convergence for Square Cavity



Convergence for Triangular Cavity, Re = 400

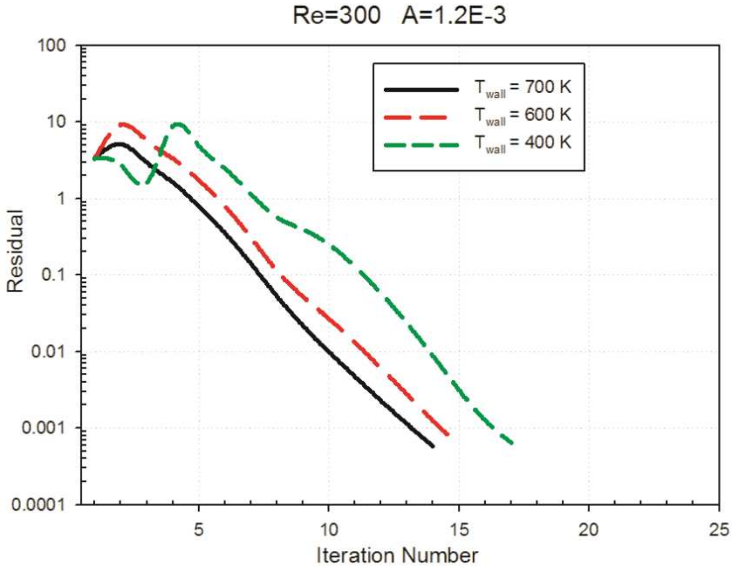


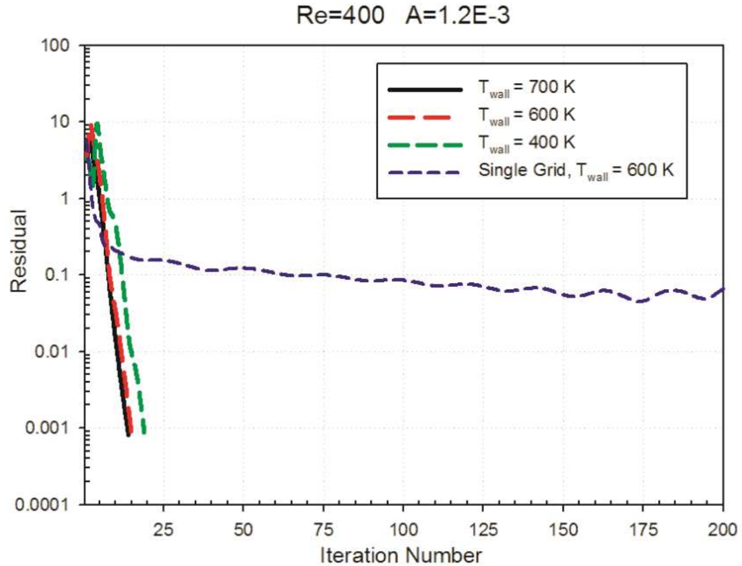
Types of Wavy Channels

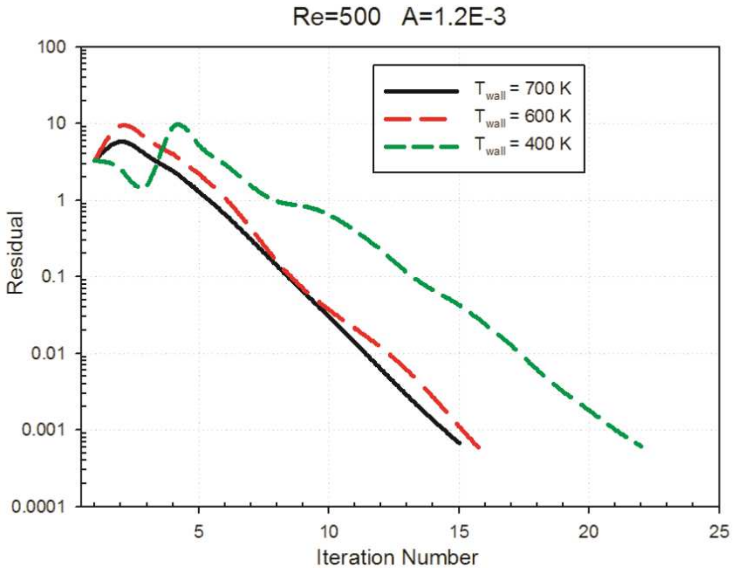


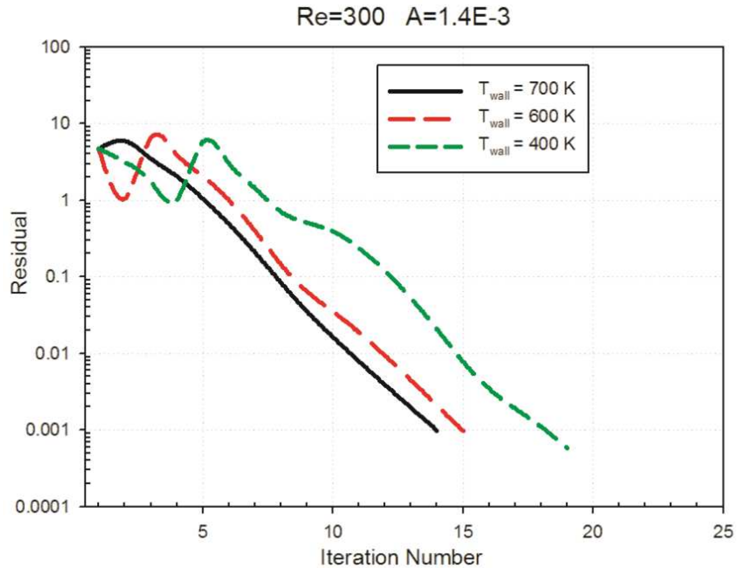
Convergence,

64 x 1280 grid, 6 waves, 4 straight sections, 4 levels



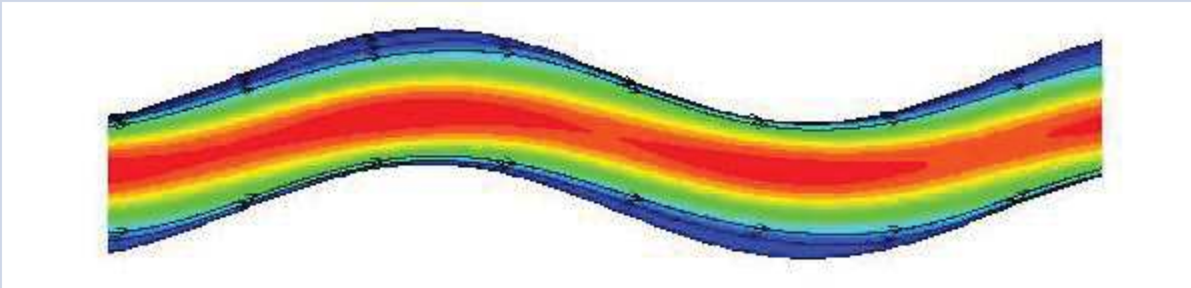




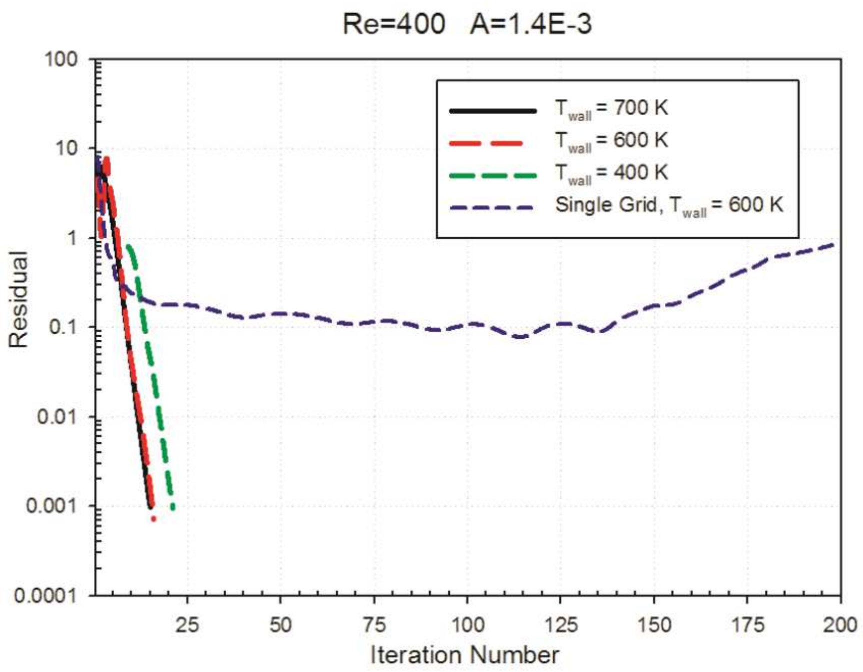


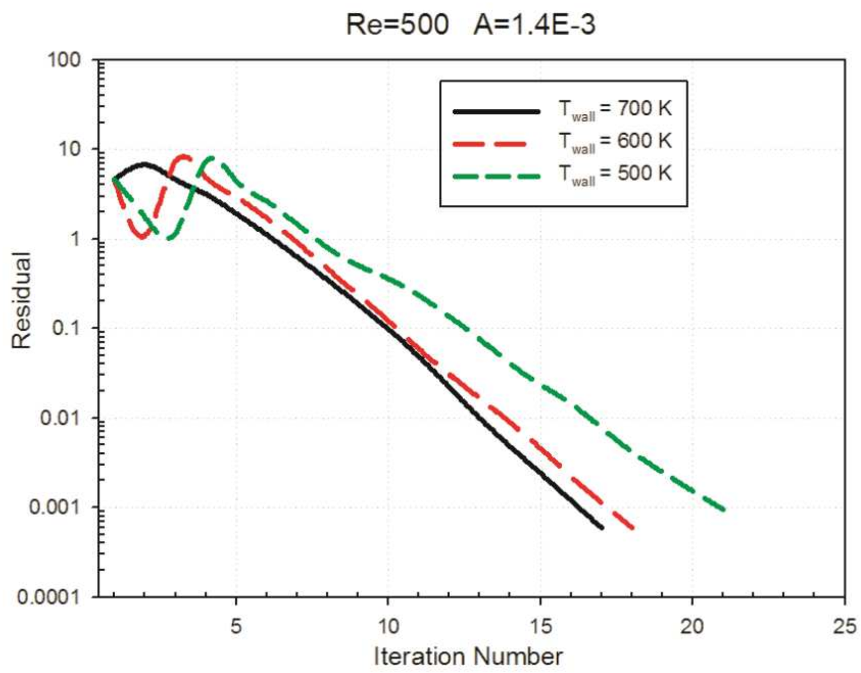
Selected Flow Patterns

Velocity Contours for Re = 300, Twall = 500, Amp = 1.2mm



Convergence





Where Multigrids are applicable

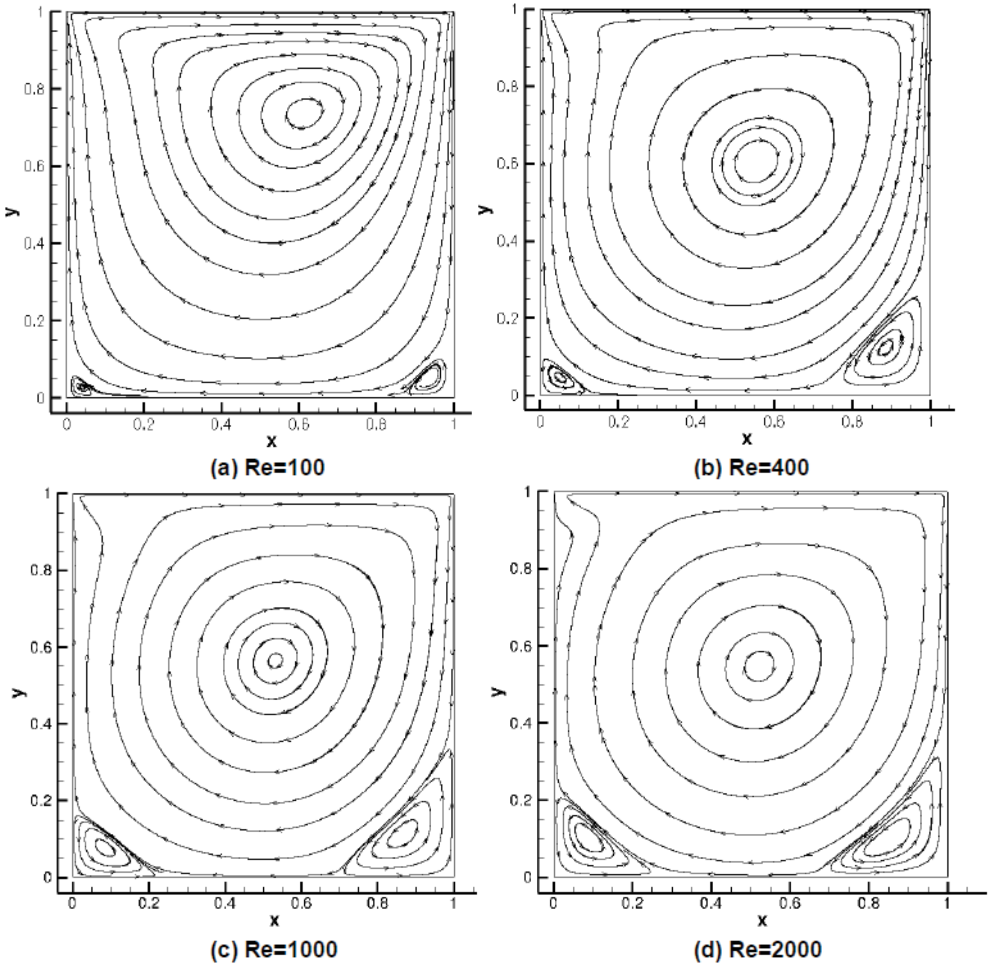
- The multigrid concept is most useful for elliptic partial differential equations.
- There is some research literature on hyperbolic and parabolic problems and on integrodifferential equations.
- For fluid flows, the multigrid technique is efficient for recirculating flows and for steady state situations.
- Multigrid techniques are applicable for heat conduction, radiation and equations in electromagnetics.

Where Multigrids may be less rewarding

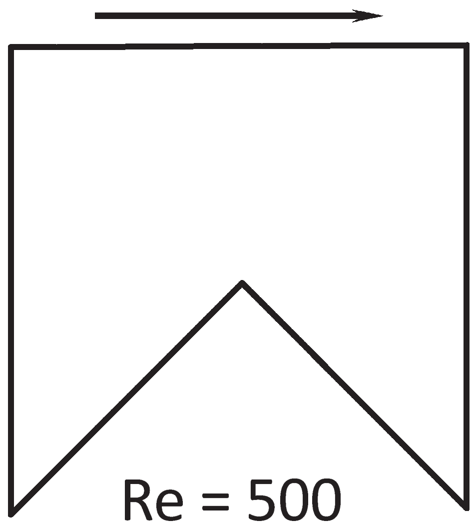
- Whenever there is not much low frequency content in the error, the cycling between fine and coarse grids is less rewarding. This is because the fine grids perform almost all the required smoothing.
- For time-dependent (and parabolic) flows the efficiency depends on the time step (or the forward step).
- For problems in which the transport operator may be less important then the source terms (chemical kinetic equations and turbulence transport equations).

EXAMPLES OF INCOMPRESSIBLE FLOW SIMULATIONS

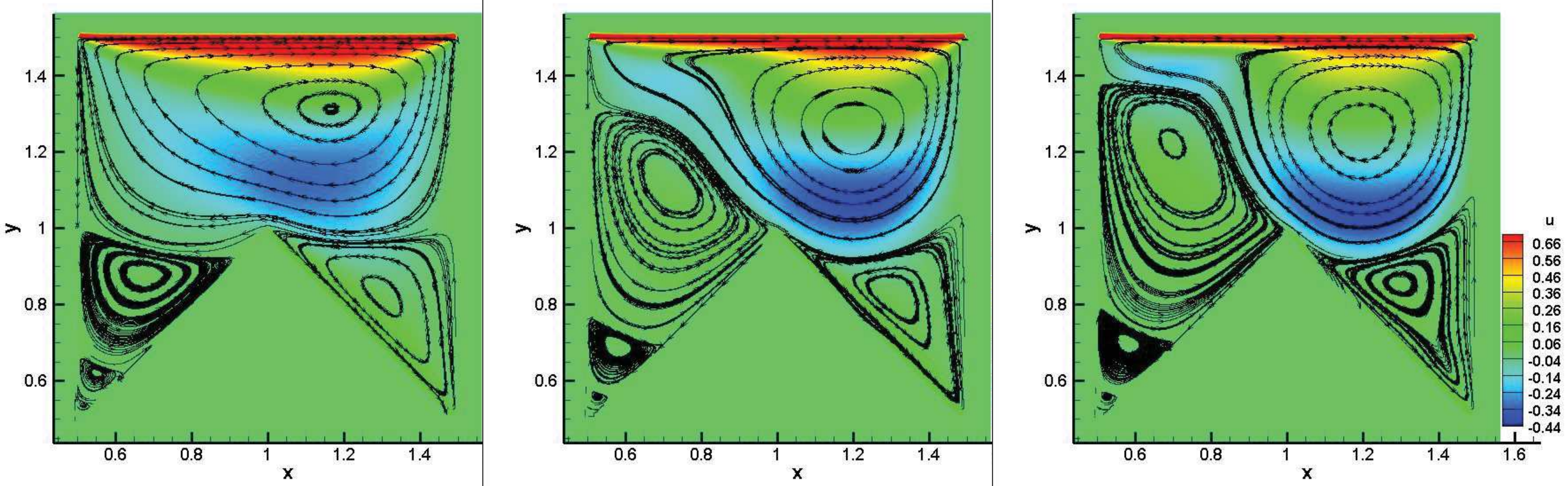
Flow in a driven Square Cavity



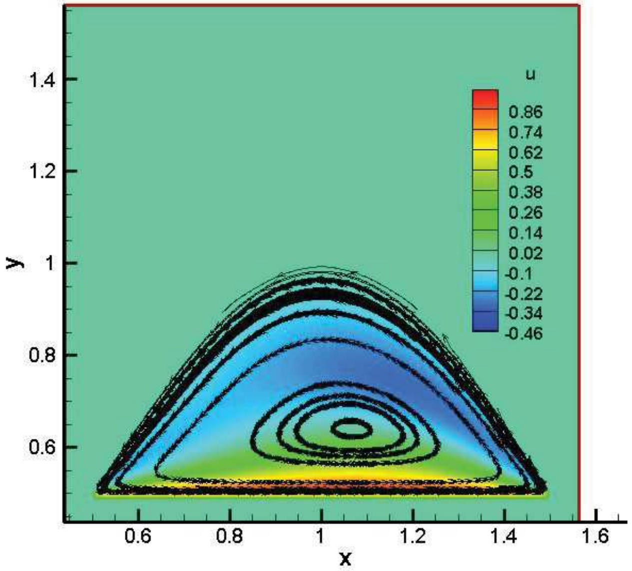
Flow in a Triangular Pedestal Cavity



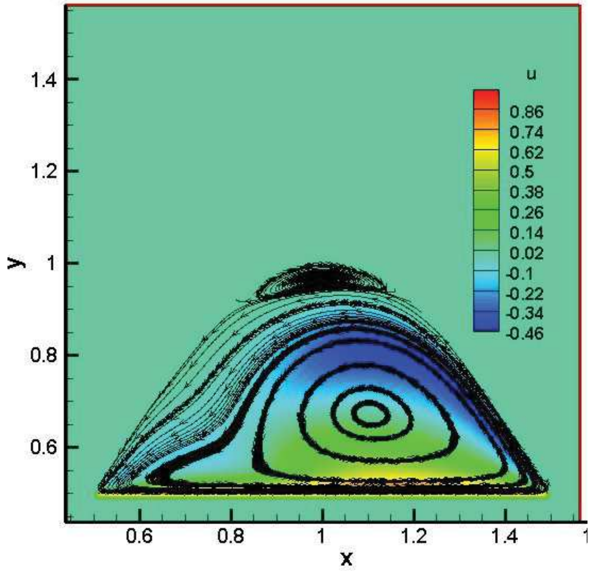
Re = 100 Re = 500 Re = 1000



Flow in a Sinusoidal Cavity

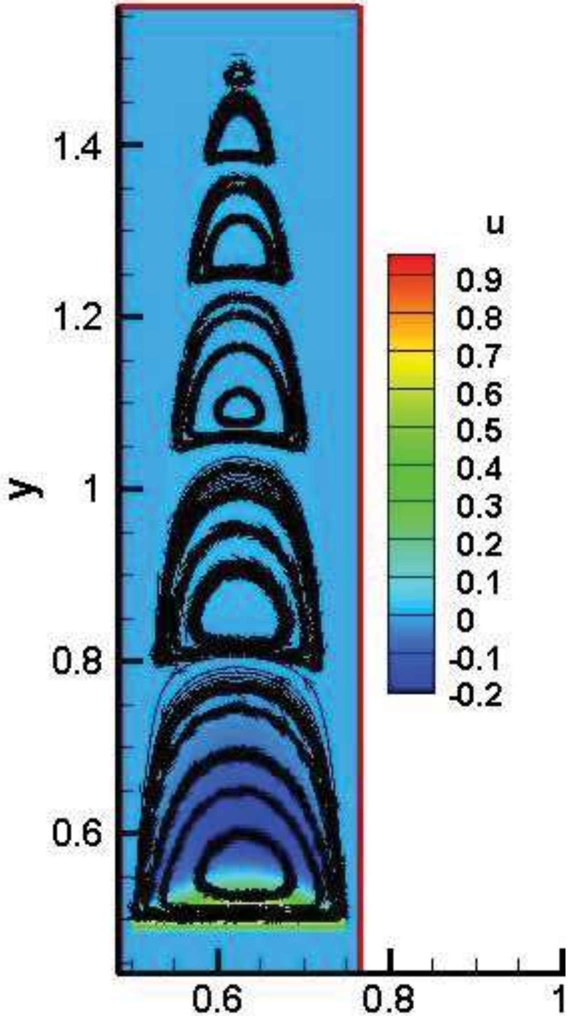


Re = 100, h = λ/2

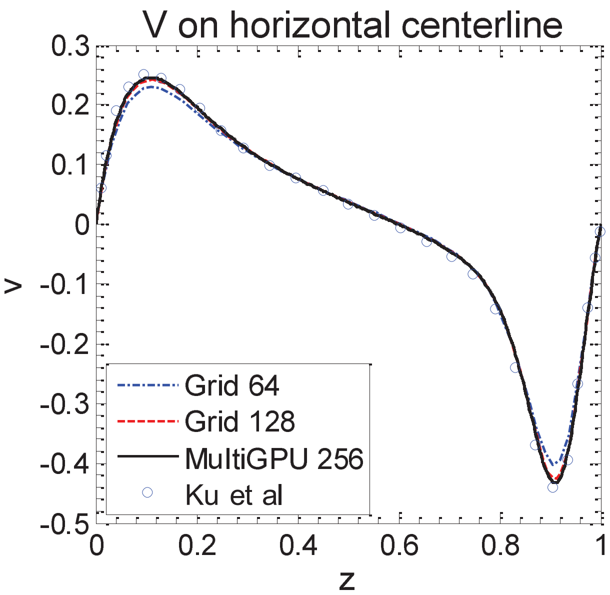


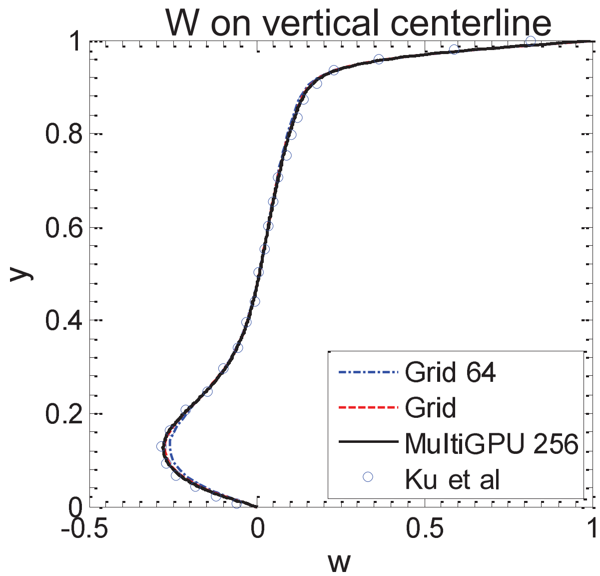
Re = 1000, h = λ/2

Re = 50, h = 2 λ



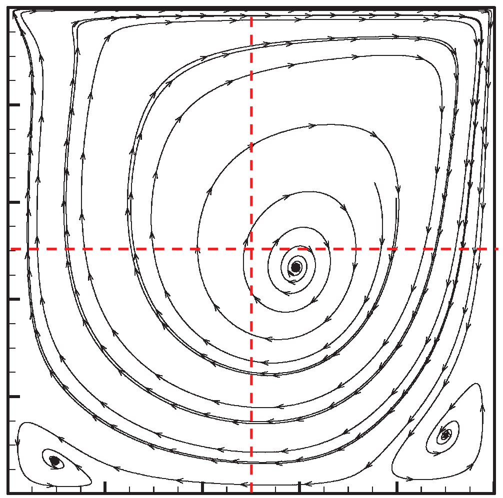
Newtonian flow in a 3D cavity with moving top lid



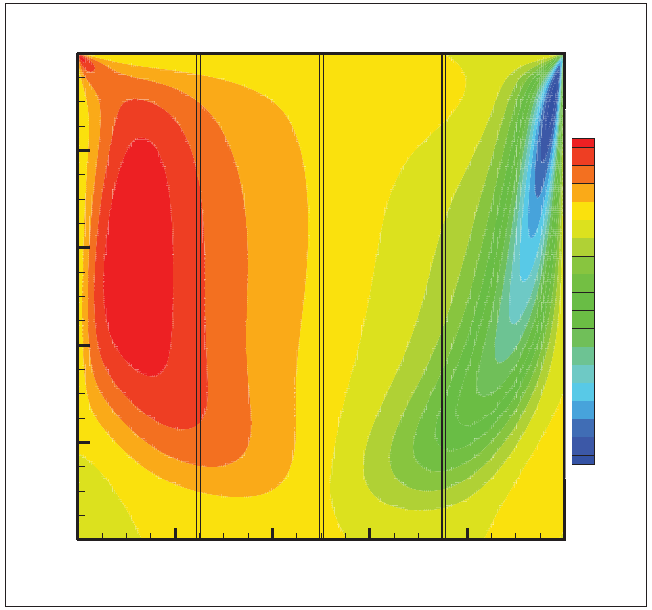


*Ku, H.S., Hirsh, R.S., and Taylor, T.D., "A Psuedo-Spectral Method for Solution of the Three-Dimensional Incompressible Navier-Stokes Equations," J. of Computational Physics, Vol. 70, 1987, p. 439.

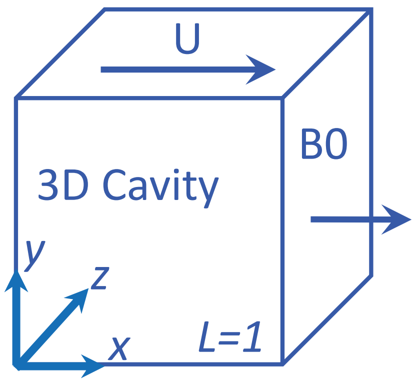
Streamlines in Symmetry Plane



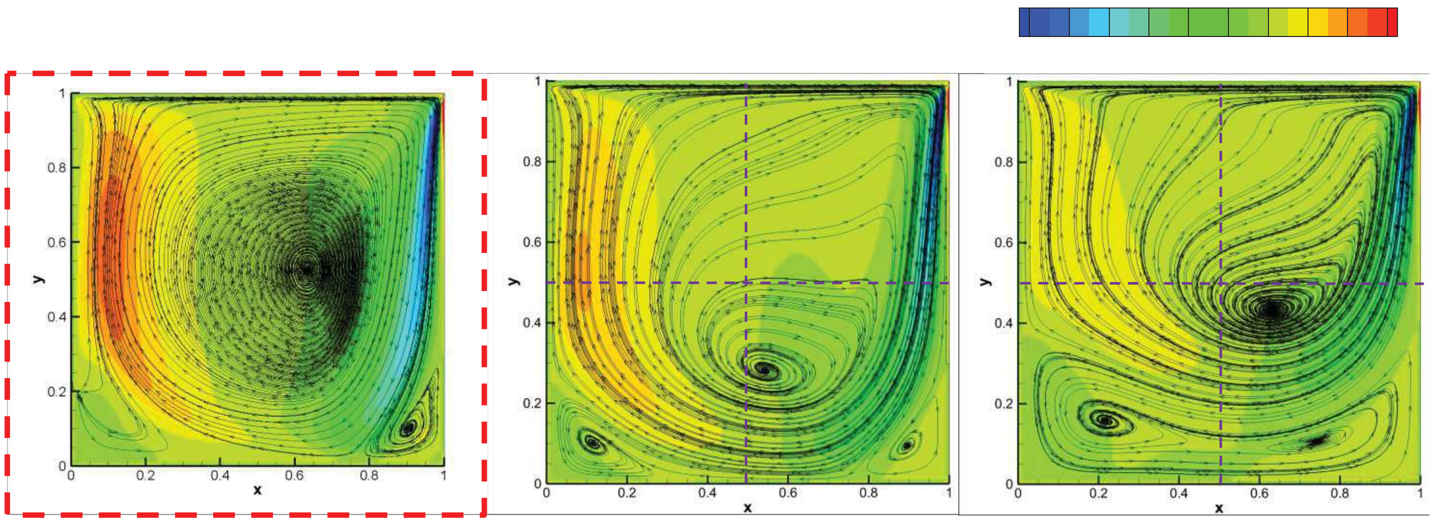
Contour of v in Symmetry Plane

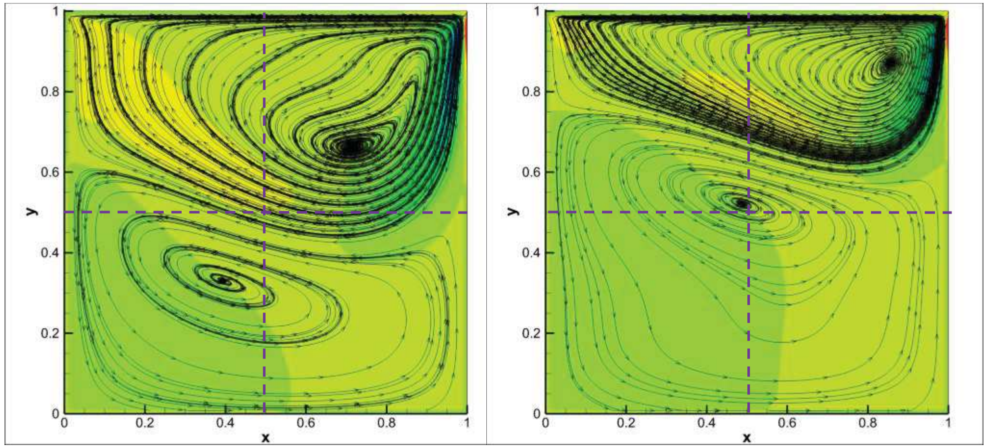


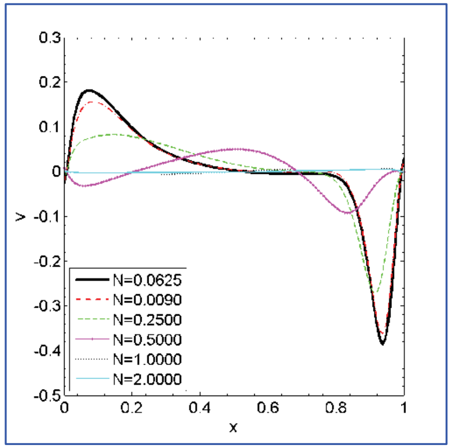
Numerical Simulation of MHD Flow in a Cube with Re = 3200



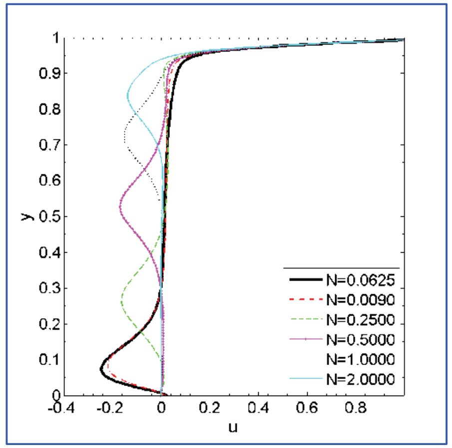
BC: Insulated & no-slip walls; Lid velocity U.





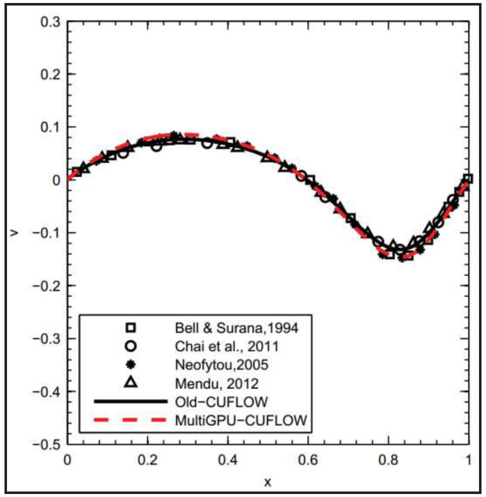


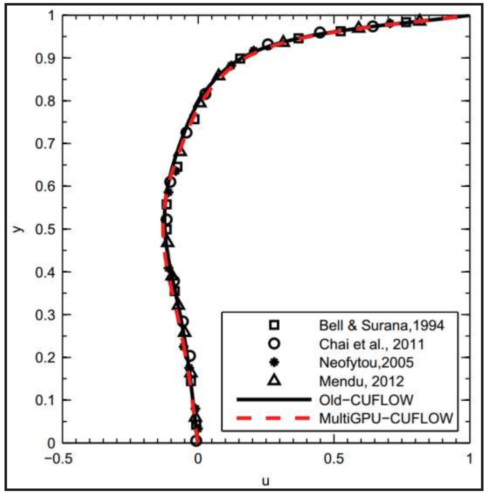
v on horizontal centerline

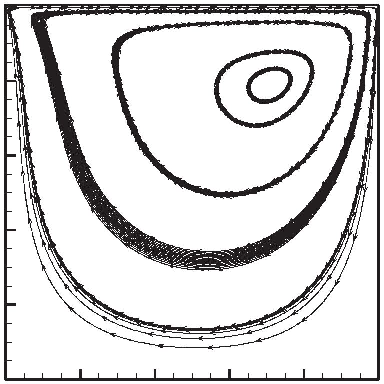


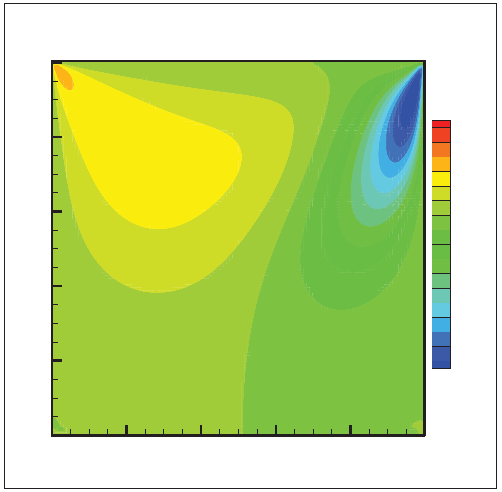
u on vertical centerline

Flow of a Shear-thinning Fluid in a long Lid-driving Cavity

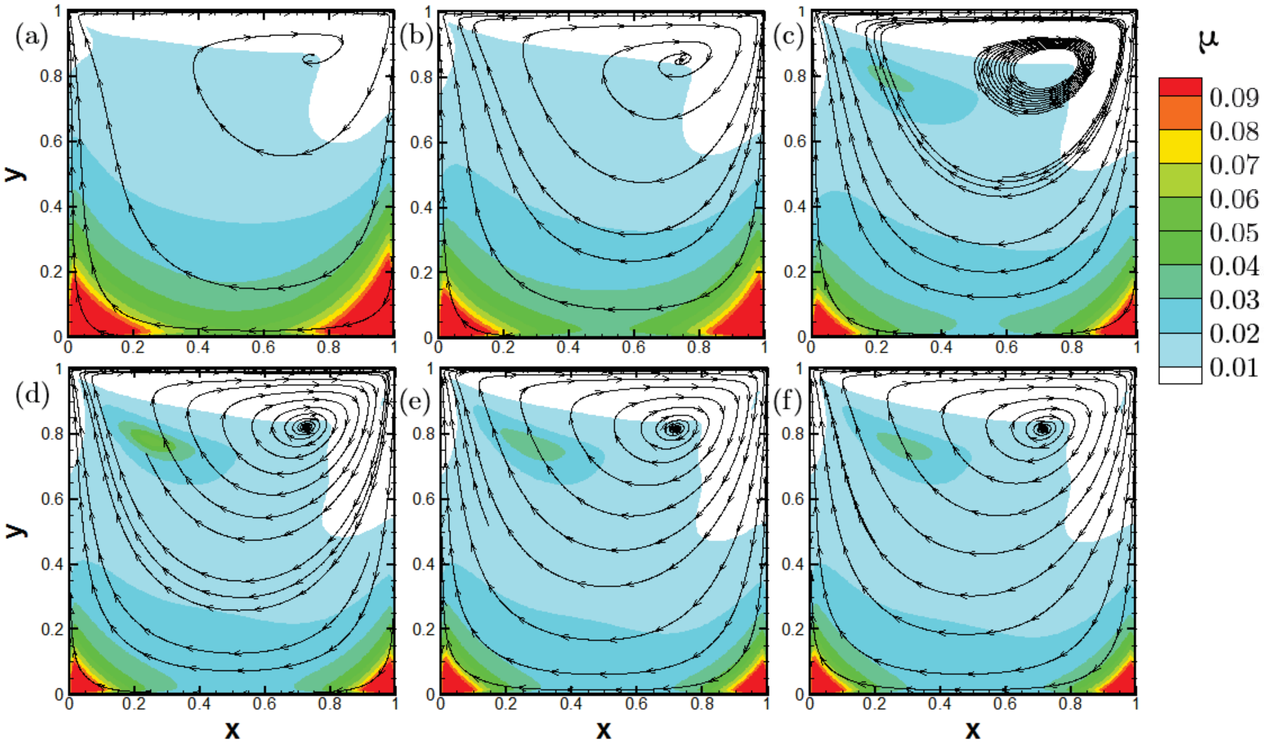


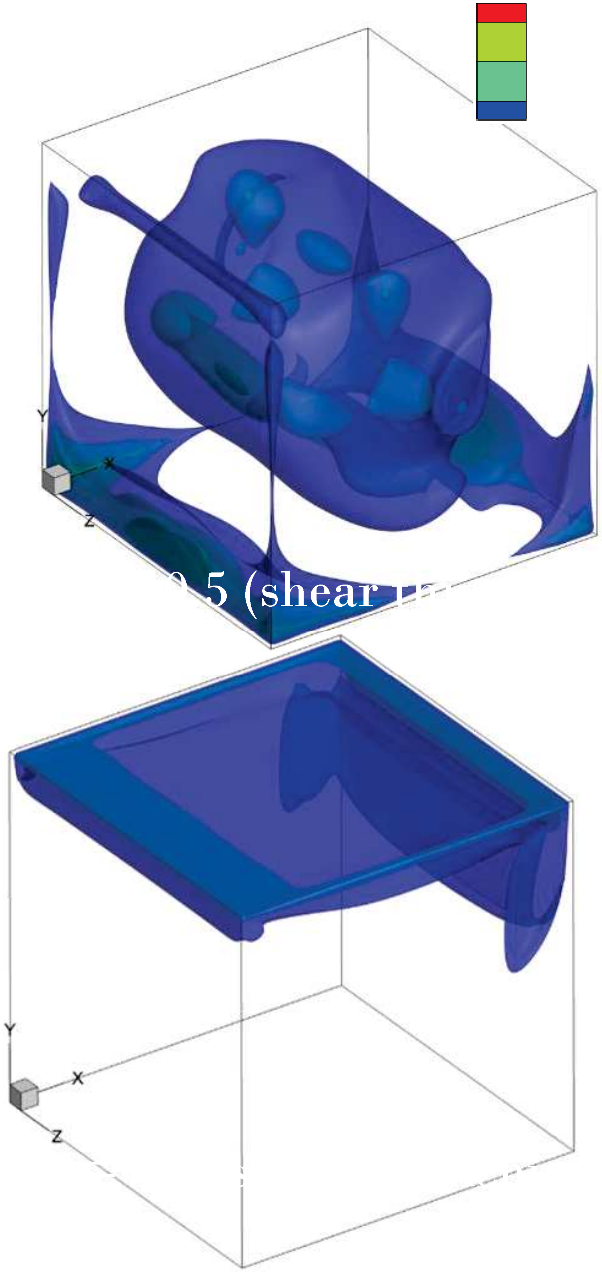




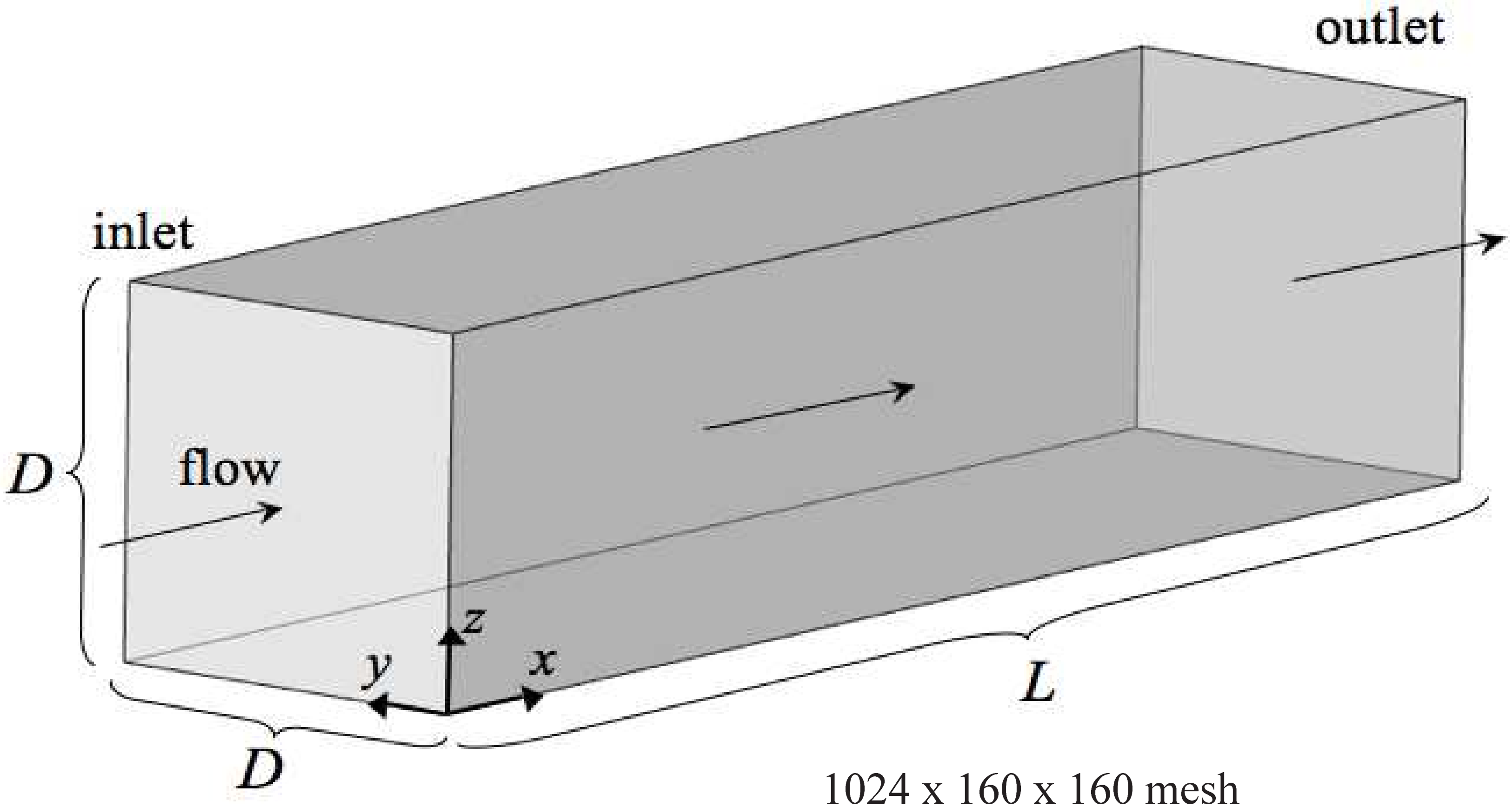


3D Cube Flow of a non-Newtonian Fluid



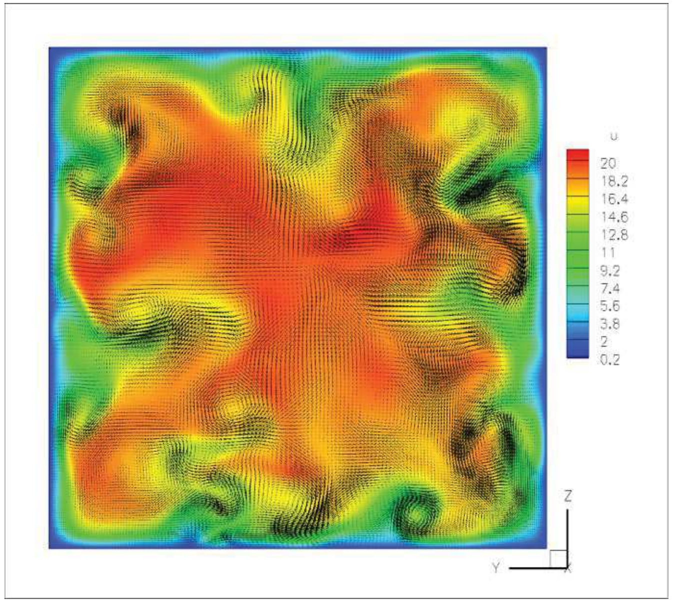


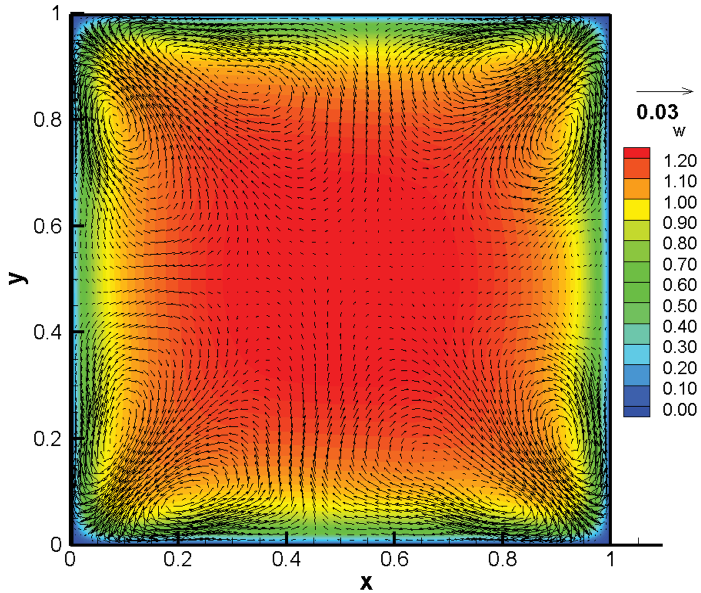
DNS of Turbulent Flow in Square Duct



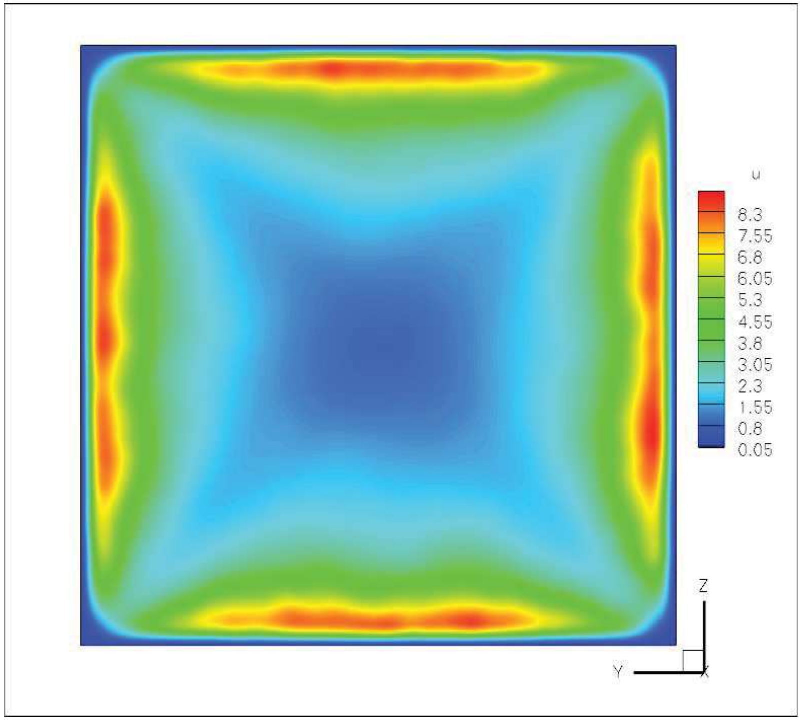
1024 x 160 x 160 mesh

DNS of Turbulent Flow in Square Duct

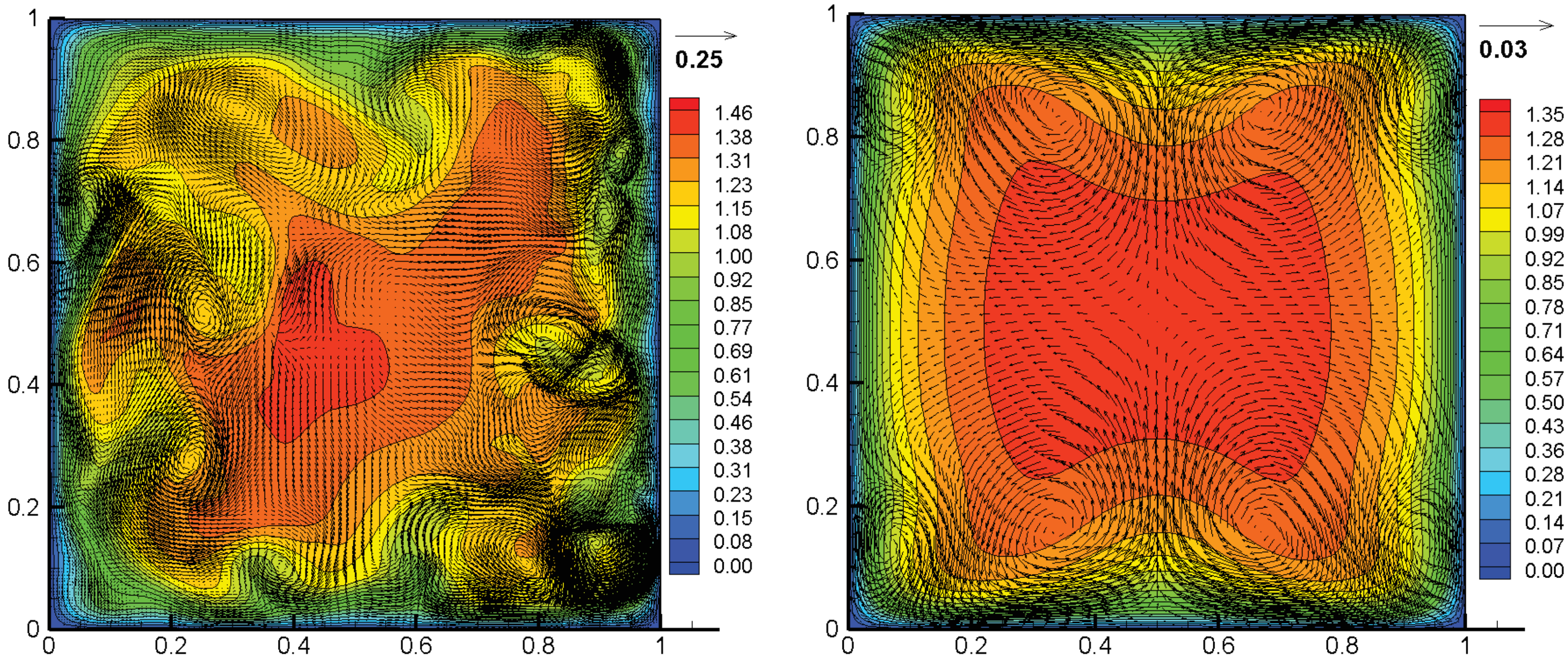




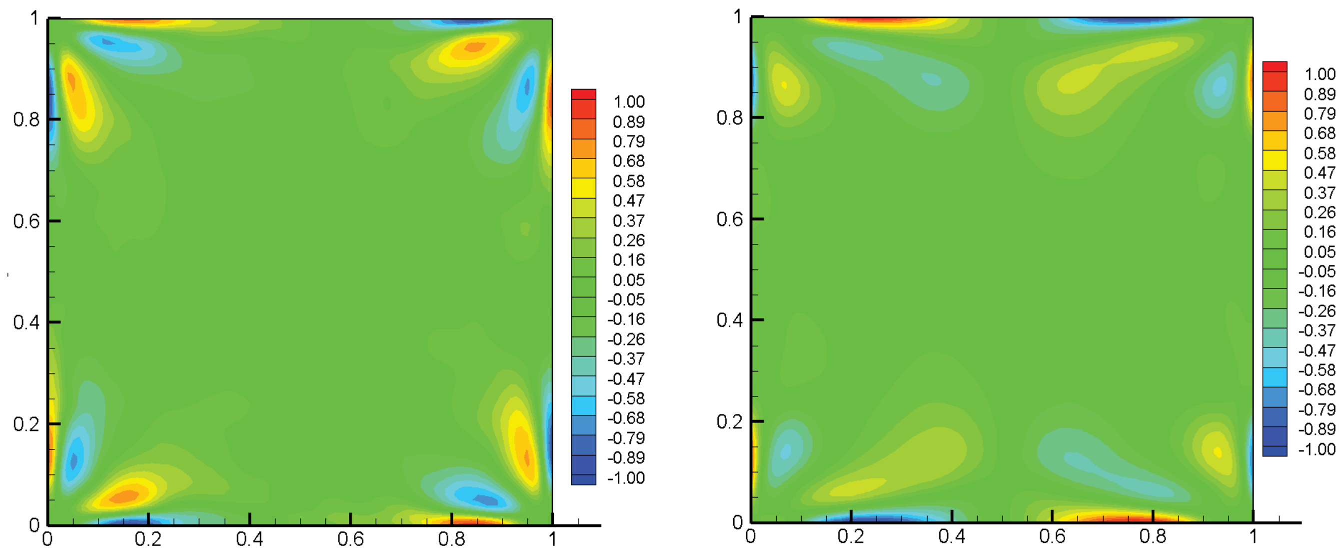
U rms



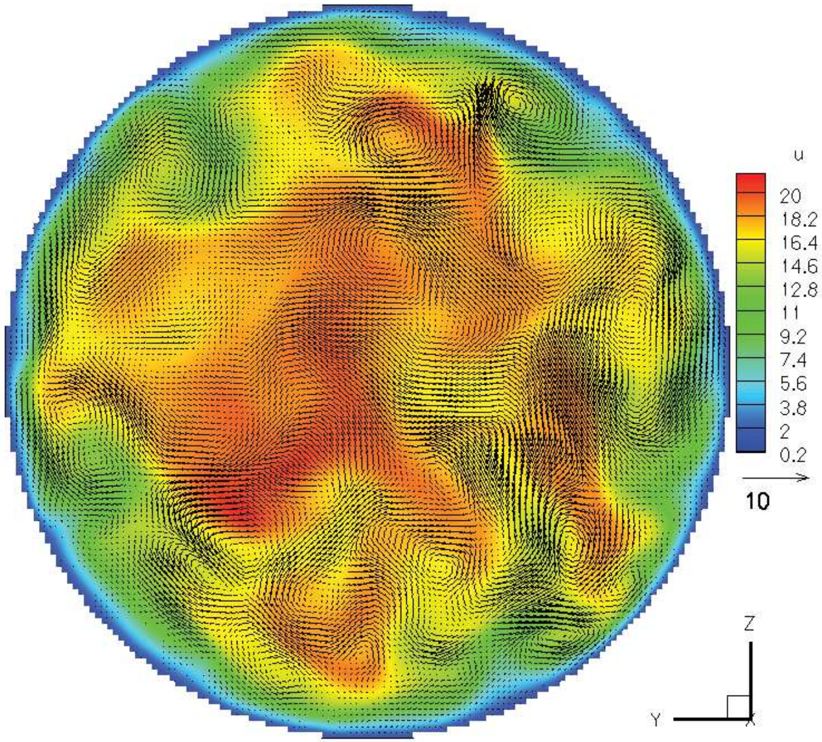
Flow pattern- with MHD

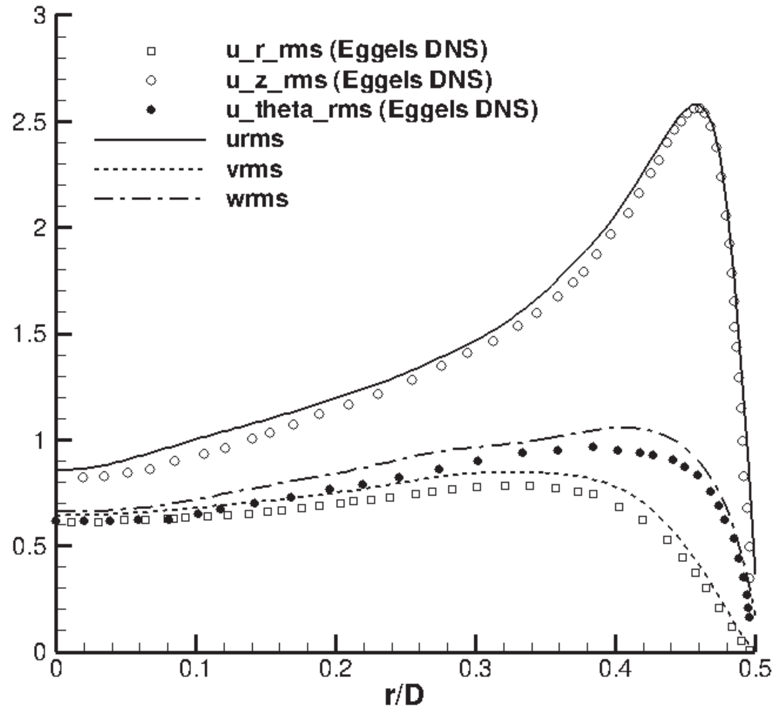


Mean streamwise vorticity

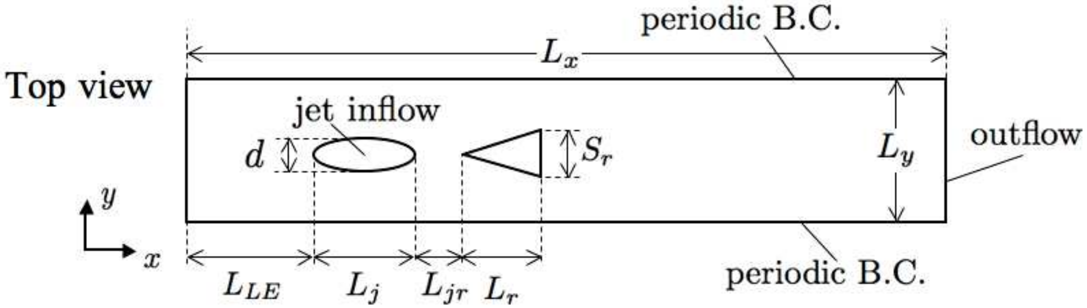


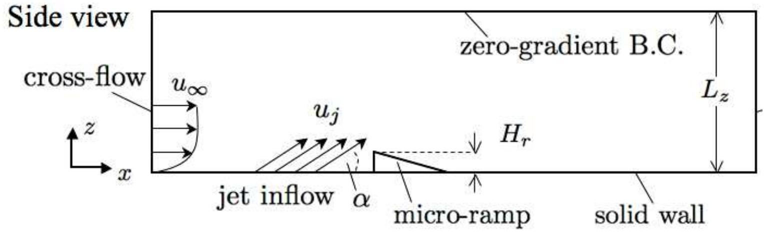
Turbulent Pipe Flow



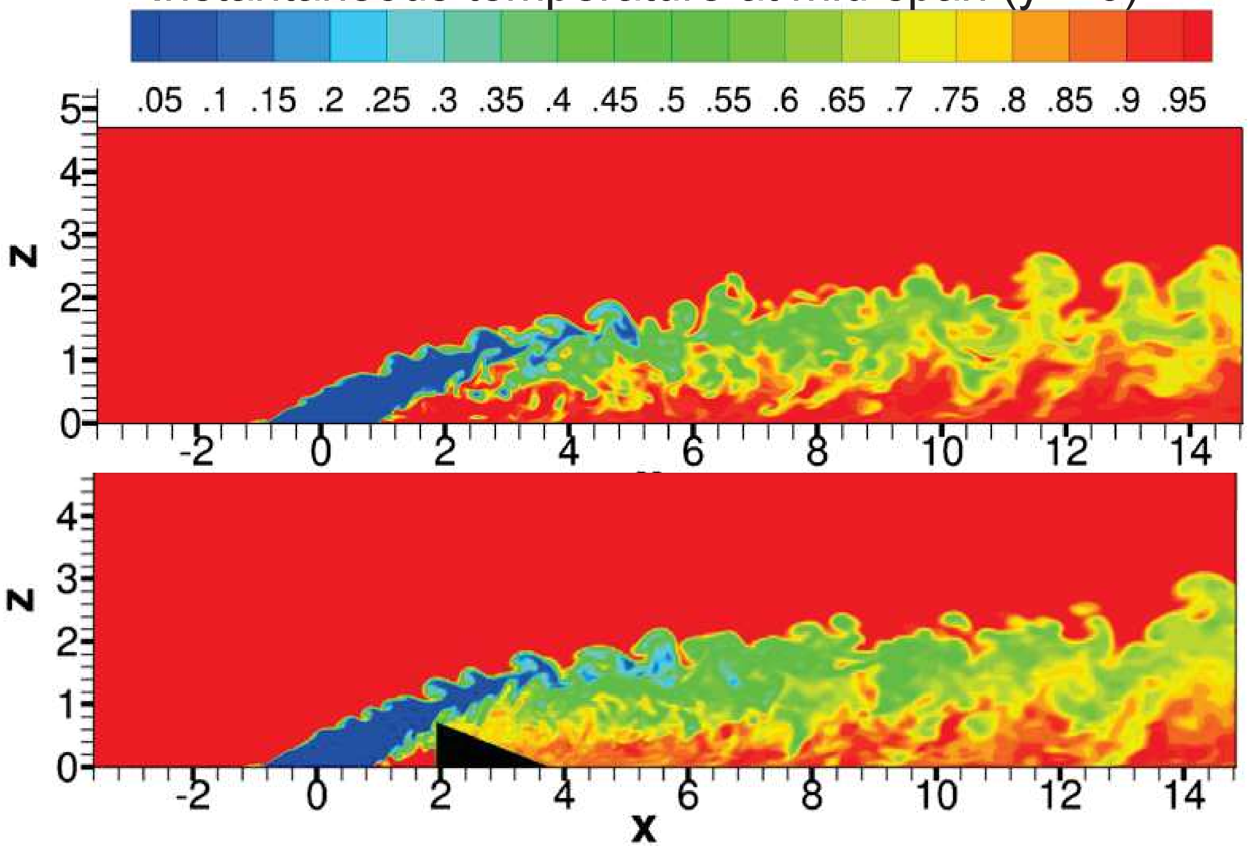


Application in Film Cooling





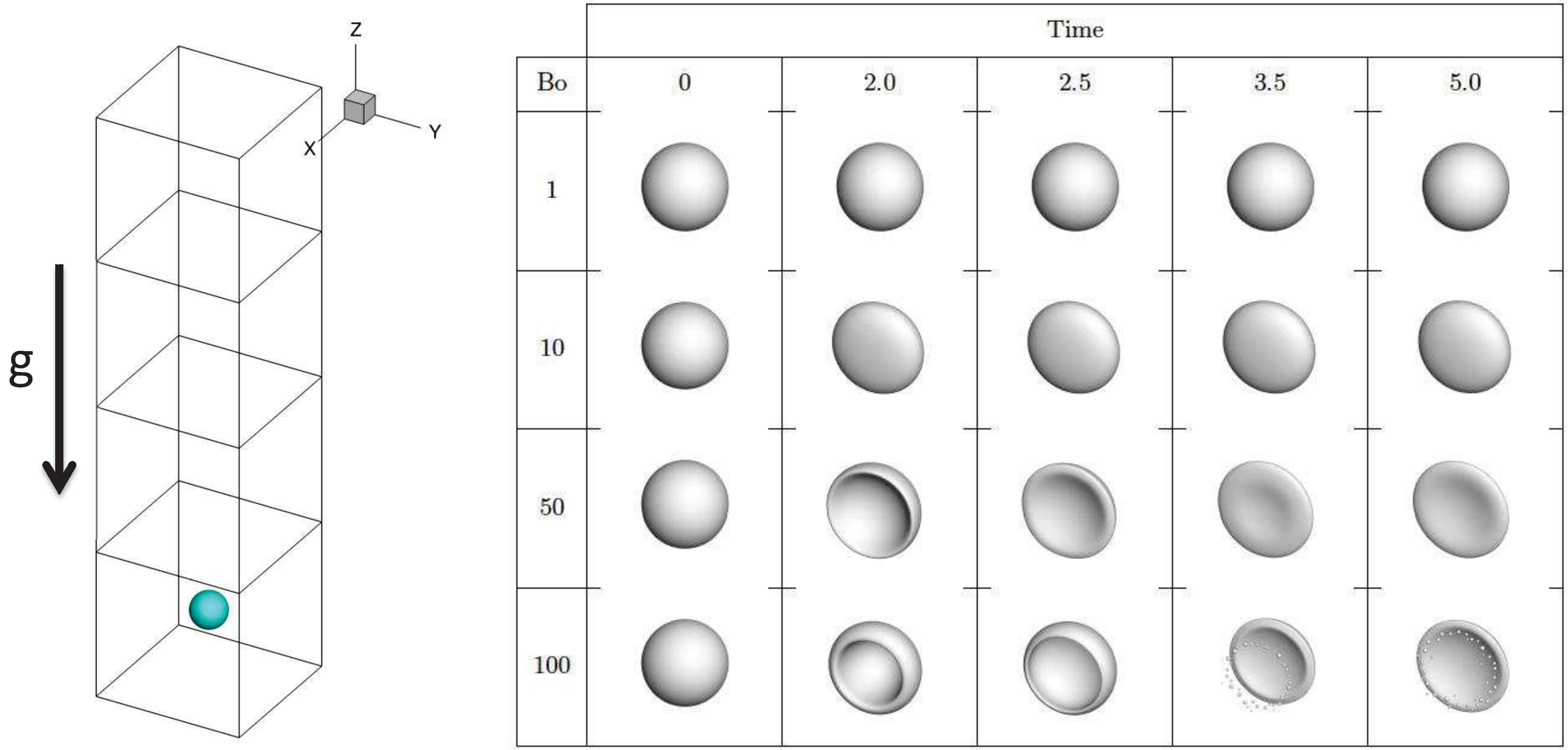
Instantaneous temperature at mid-span (y = 0)



Dimensions for computational domain

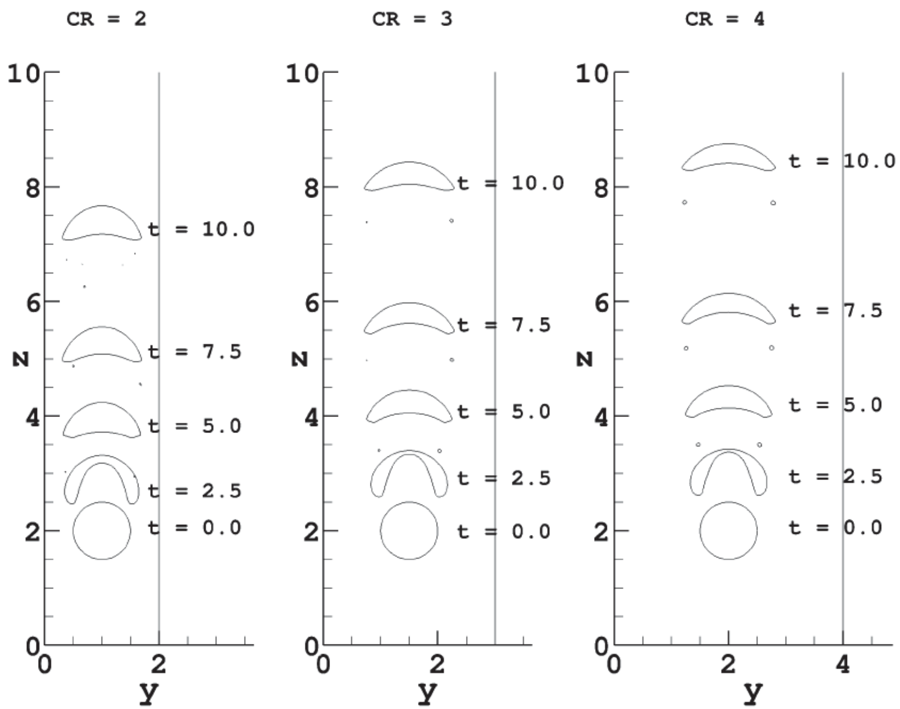
L_x	$18.4d$
L_y	$3.0d$
L_z	$4.7d$
L_{LE}	$2.7d$
L_j	$1.743d$
L_{jr}	$1.0d$
L_r	$1.913d$
S_r	$1.573d$
H_r	$0.748d$
α	35 degrees

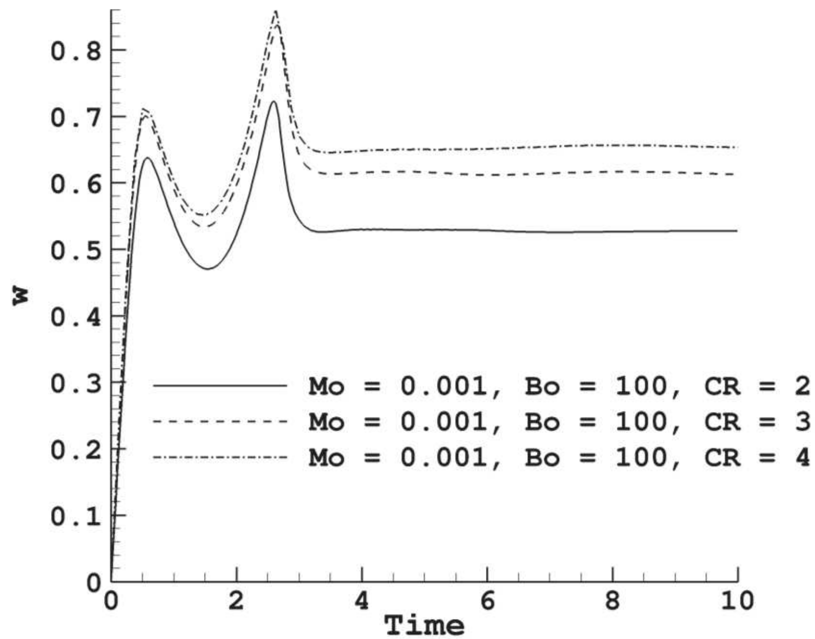
Bubble rise in a liquid column: effect of confinement



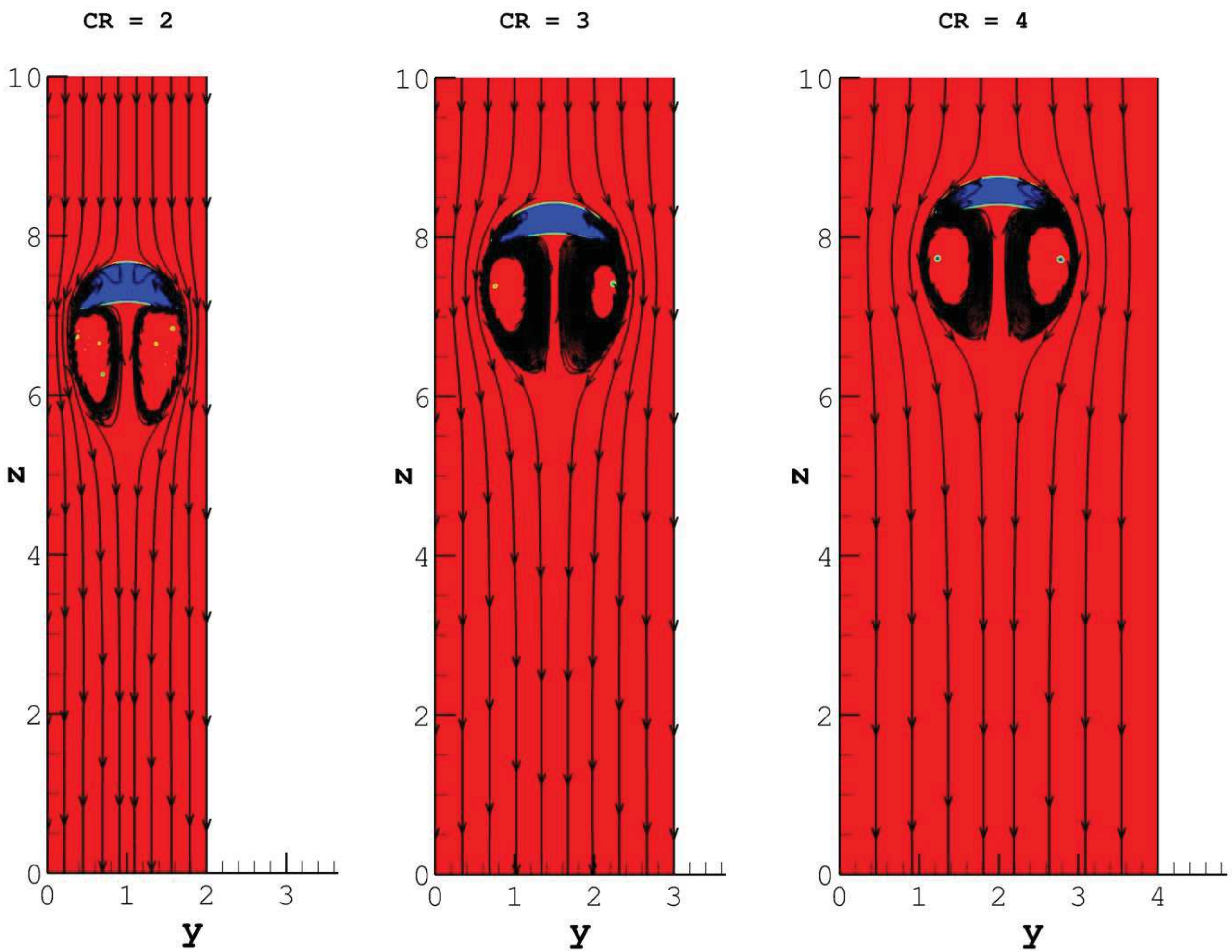
Transient bubble shapes for confinement ratio of 4 and Mo = 0.001

Effect of confinement

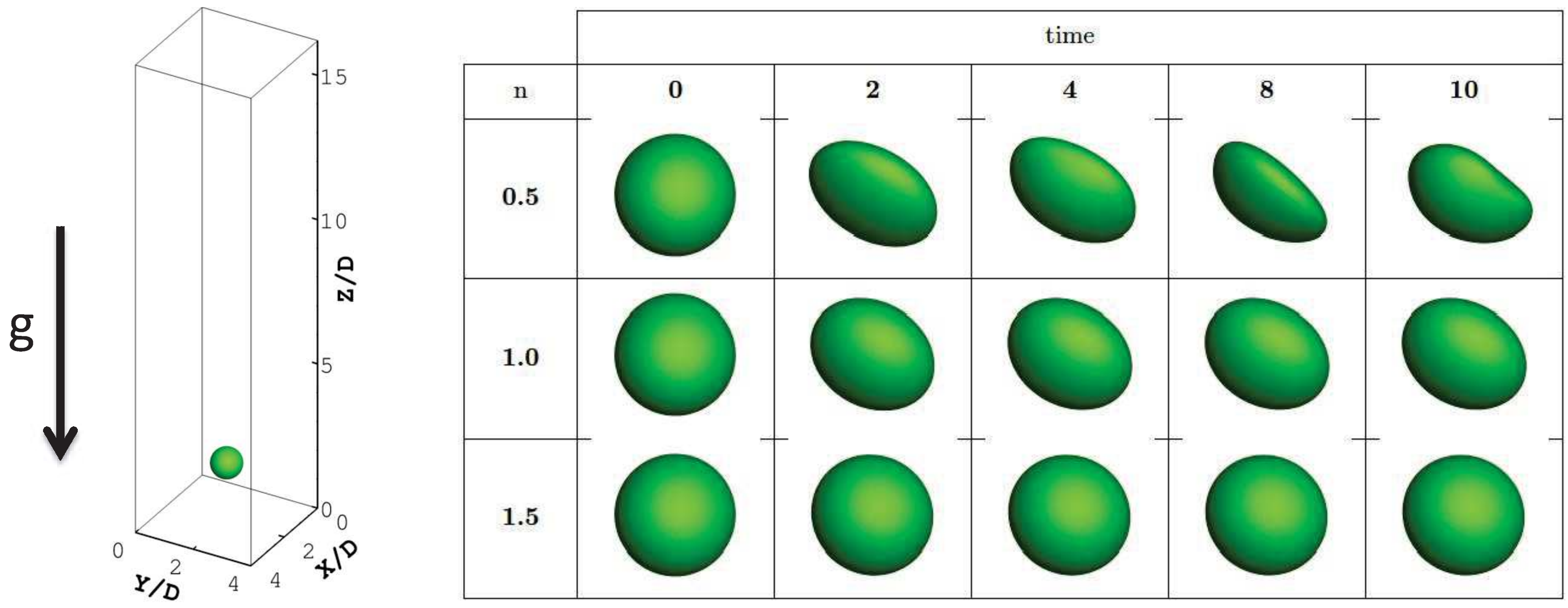




Effect of confinement

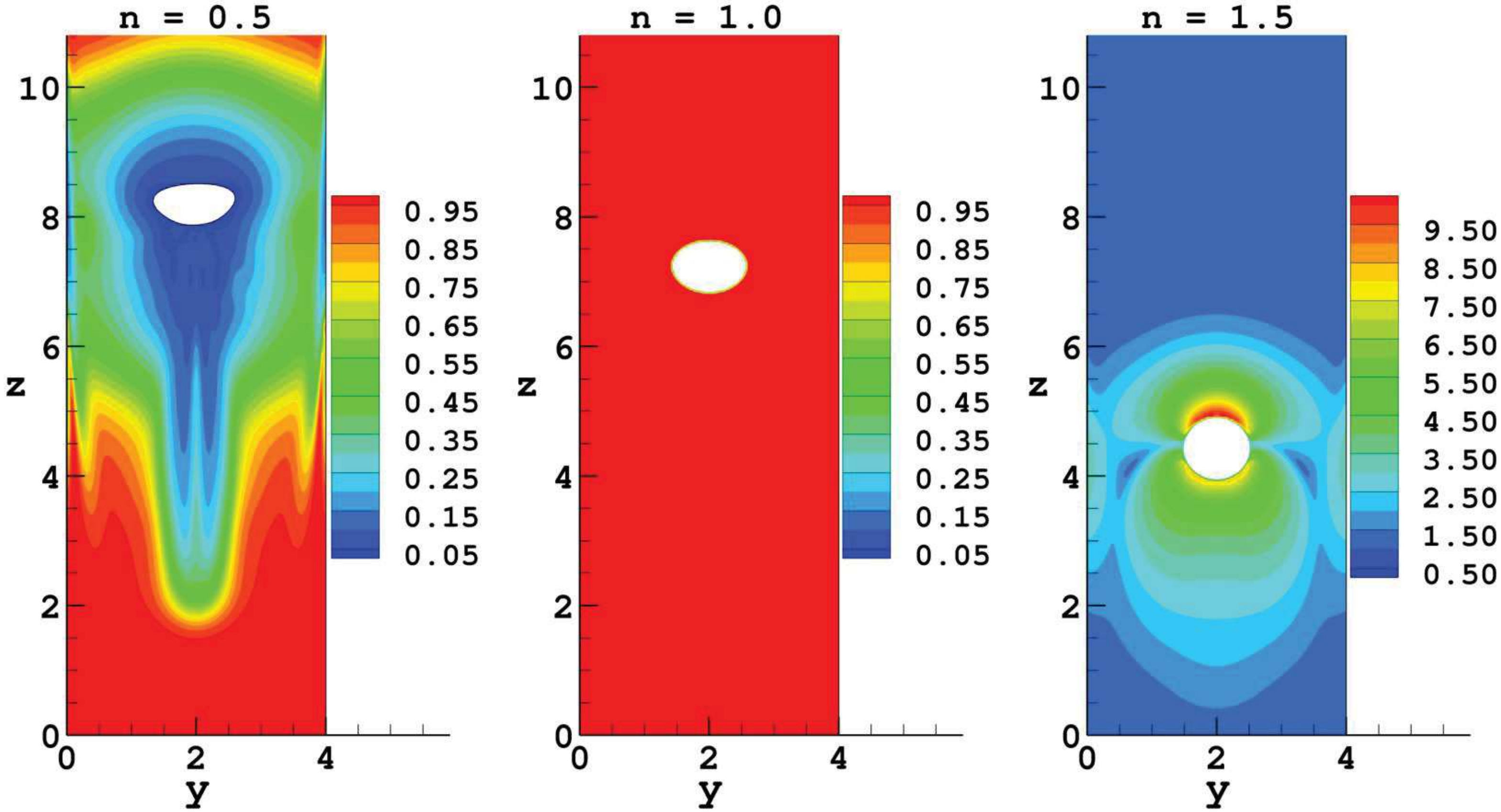


Bubble dynamics in non-Newtonian liquids



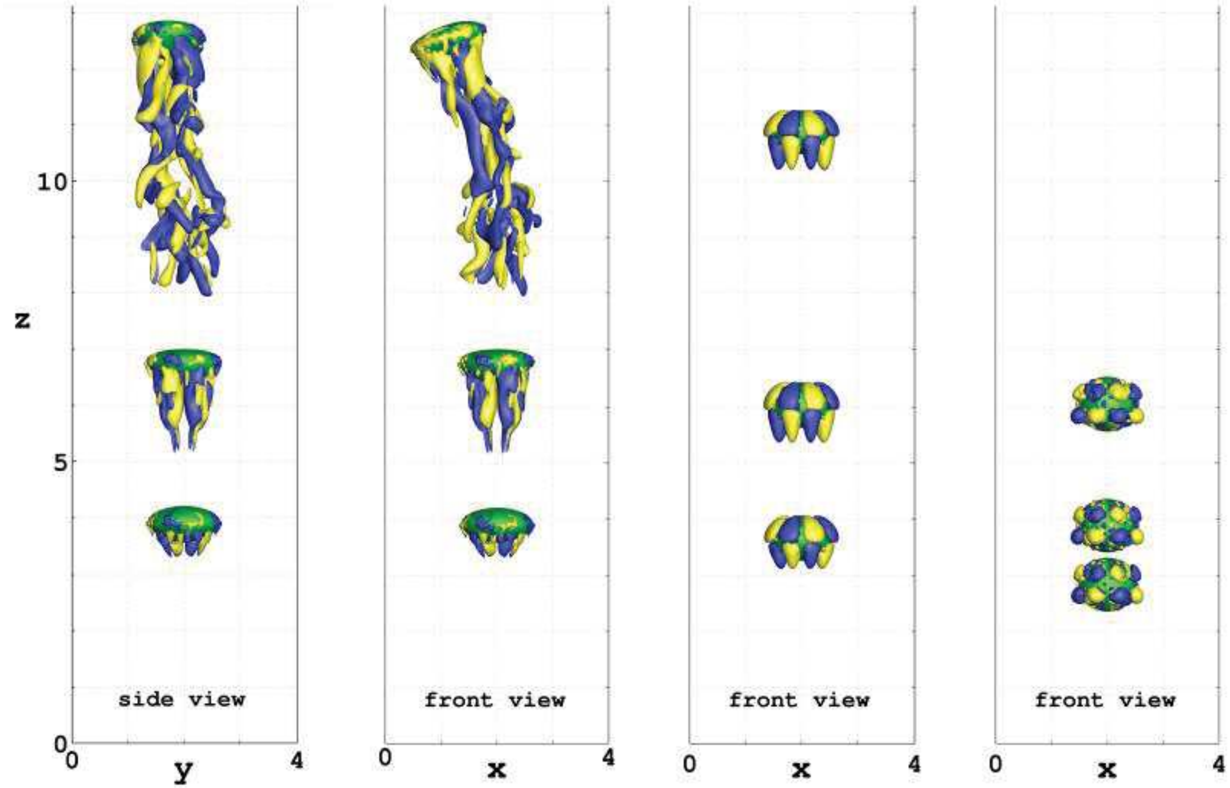
3D perspective view of the bubbles at different times for Bo = 2 , Mo = 1E-06, in shear-thinning, Newtonian and shear-thickening fluids.

Liquid dynamic viscosity



Liquid's dynamics viscosity contours at t = 5 on the yz center plane for *Bo* = 2 , *Mo* = 1E-06 and in shear-thinning, Newtonian and shear-thickening fluids.

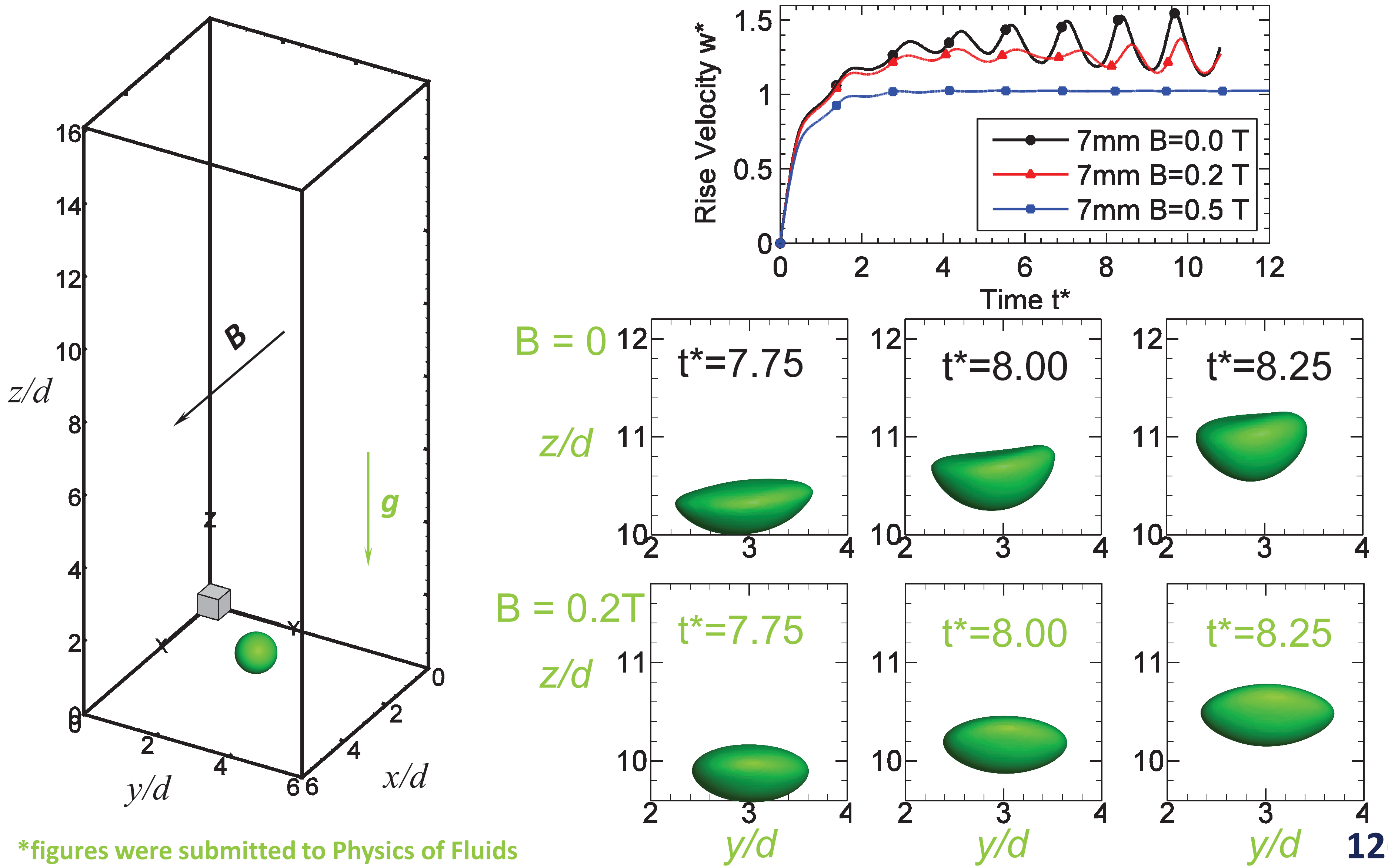
Wakes and vorticity



(a) (b) (c) (d)

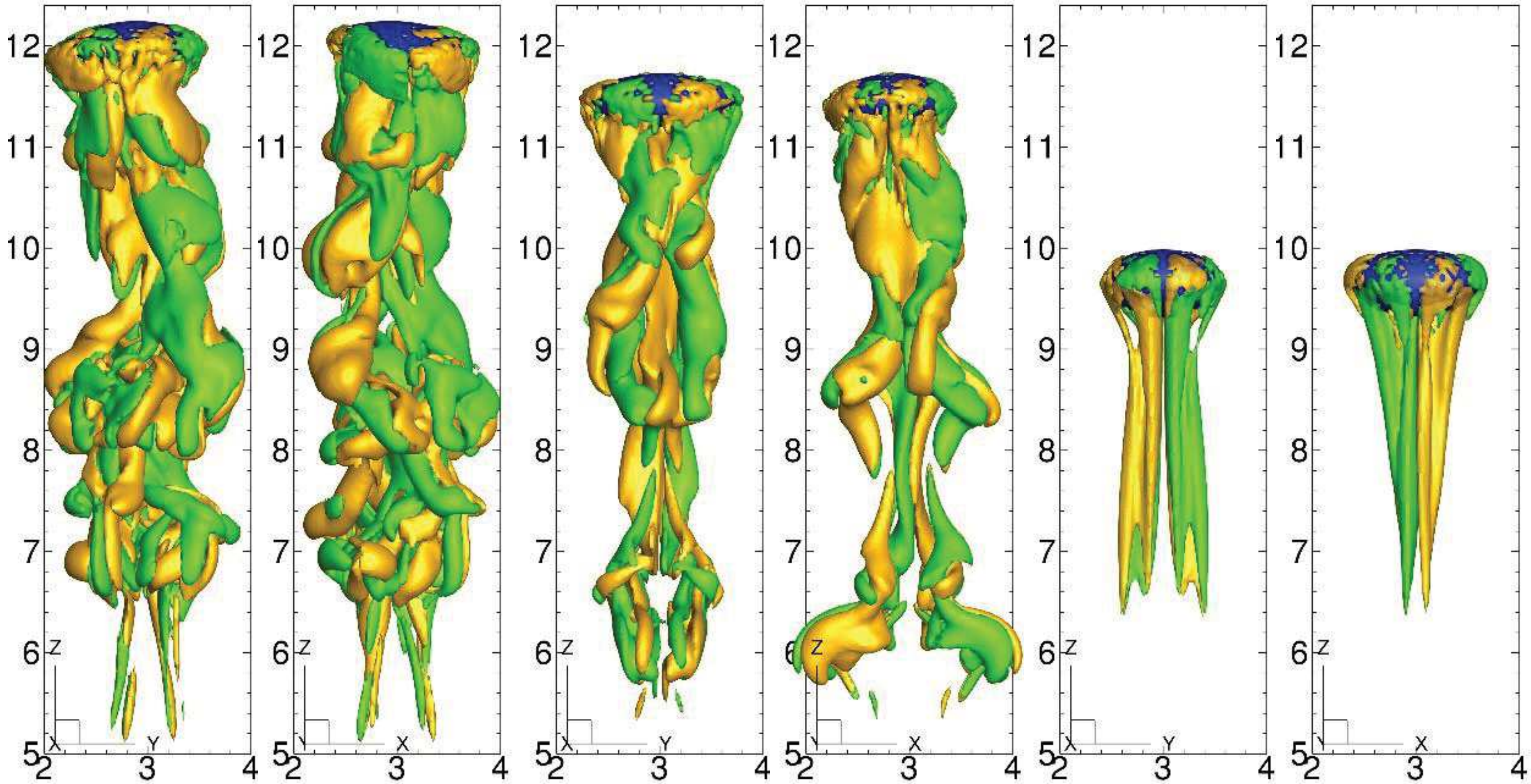
Iso-surface of z-vorticity at t = 2, 4 and 8 for shear thinning, Newtonian and shear thickening fluids. The yellow surface is 10% of maximum vorticity, the blue surface is 10% of minimum vorticity and the green surface is bubble.

Ar Bubble rise in molten steel with transverse magnetic field



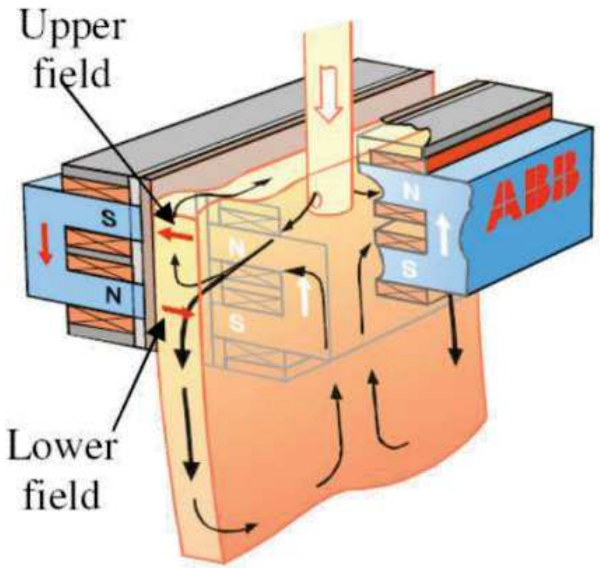
*figures were submitted to Physics of Fluids

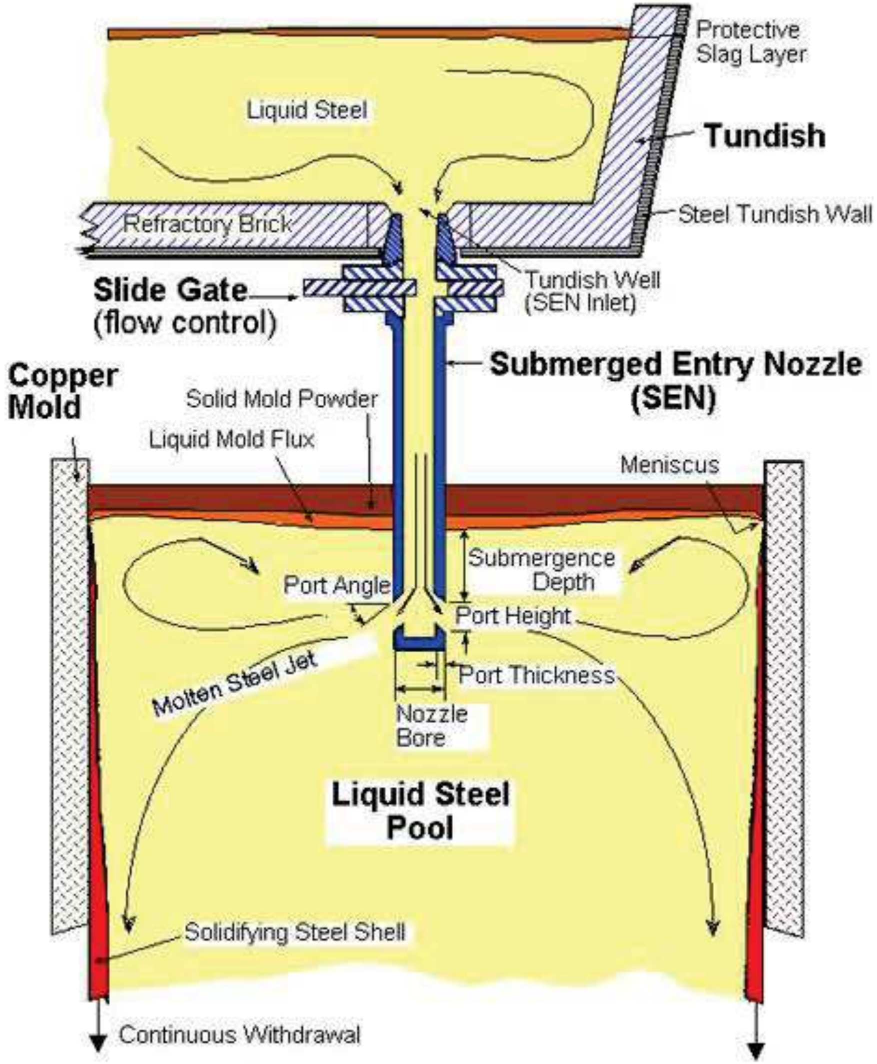
Ar Bubble rise in molten steel with transverse magnetic field



LES simulations of Effect of EMBr on Fluid Flow in the Mold at different Casing Speed

Properties of Steel	
Density (kg/m³)	7,000
Dynamic Viscosity (kg/m-s)	0.0063
Electrical Conductivity (S/m)	714,000
Magnetic Permeability (h/m)	1.26×10^{-6}





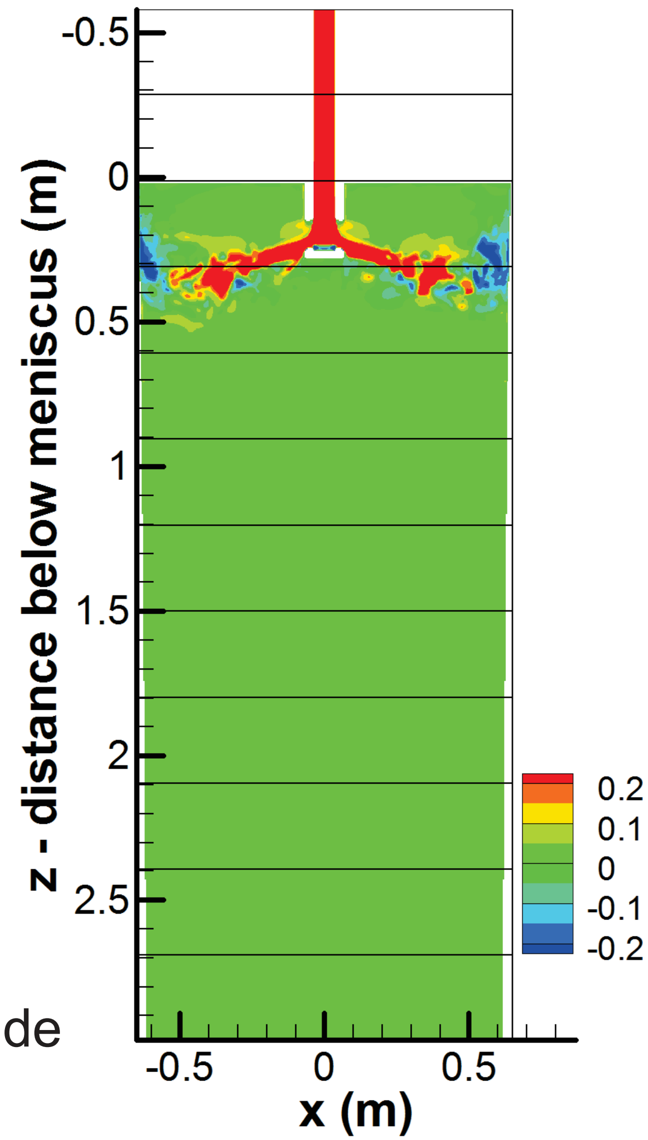
Geometry, mesh and BCs

- Full-mold domain with SEN: 15.6 million cells;

	Dimension (mm)
Lx – Mold Thickness	230
Ly – Mold Width	1300
Lz – Domain Length	6000

- Boundary Conditions:
  - SEN top: velocity inlet;
  - Bottom: zero derivative of velocity;
  - WF and NF (shell included in domain as solid):
    - Moving downward with casting speed;
    - Mass sink added
  - Top surface and walls: no penetration and no slip.

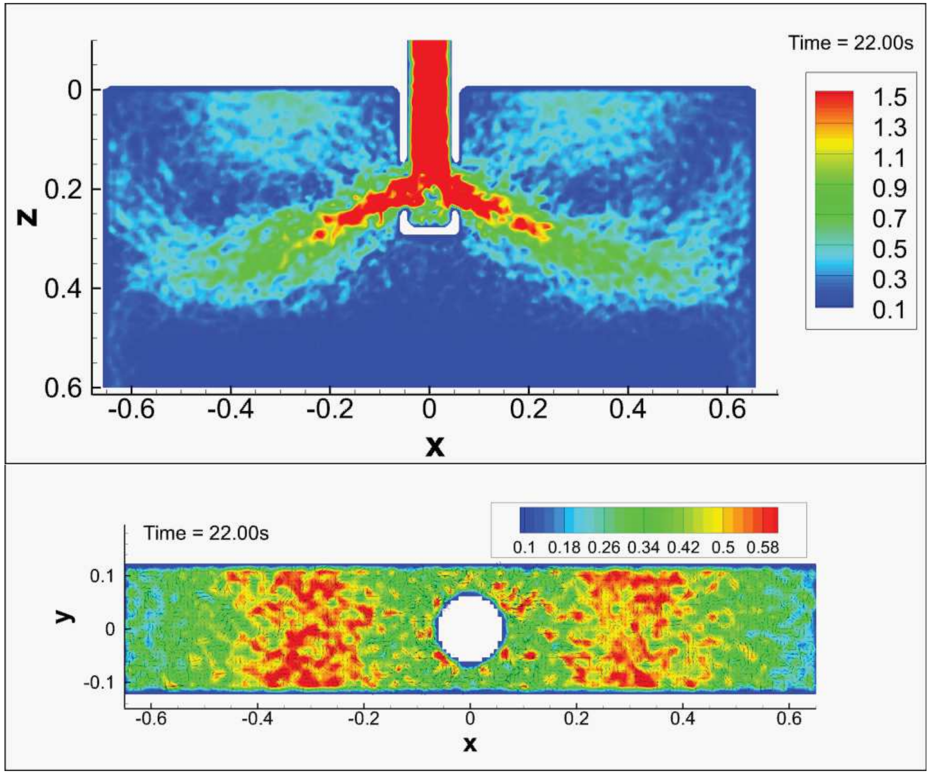
- Note the SEN inlet area is the “eye-shaped” intersected region of two circles to include the slide gate.



Effect of EMBr on Transient flow in Mid and Top Surfaces (160mm Submergence, Vc = 1.8m/min)

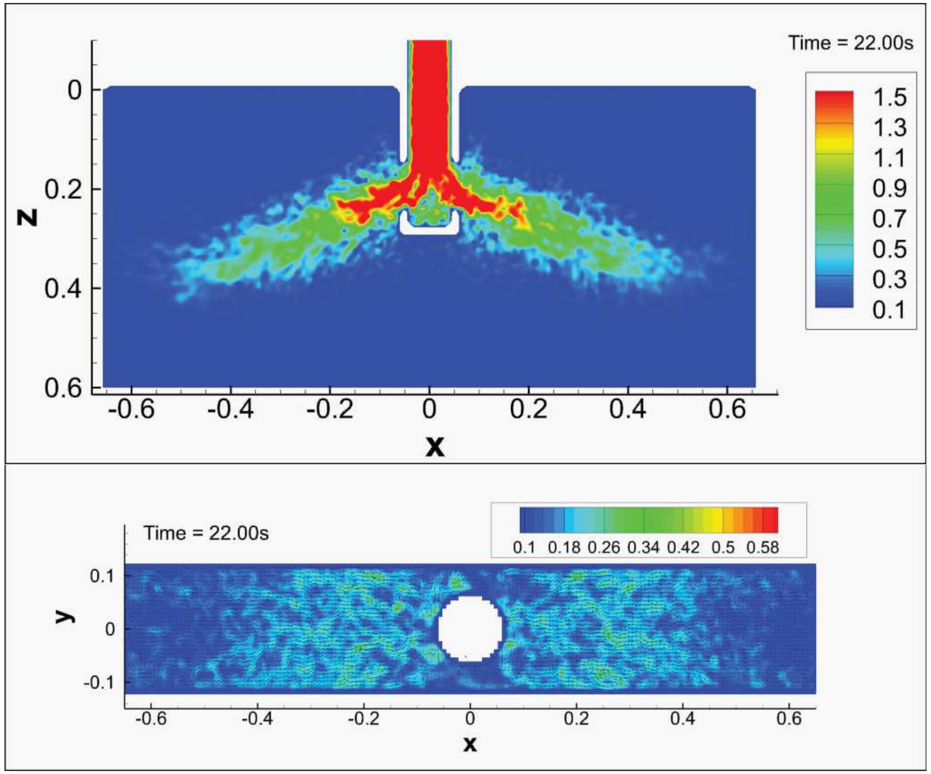
- Transient animation, contour of velocity magnitude (m/s)
- Top surface velocity reduced by **~67%** with bottom coil current 850A (B850)

VC1.8m/min-SD160mm-*NoEMBr*



Time averaged Max top surface velocity: **~0.51m/s**

VC1.8m/min-SD160mm-*B850*



Time averaged Max top surface velocity: **~0.17m/s**

High Order Finite Differencing

High Order Finite Differencing

Higher order accurate stencils can be derived by Taylor series expansions

A general procedure for stencil derivation is as follows

- Decide first the points that must be included in the stencil. These can be symmetrical or unsymmetrical, biased in desired direction. Boundaries, flow direction are two main reasons for unsymmetrical stencils.
- Write Taylor series expansions for the discrete values at the chosen stencil points.

High Order Finite Differencing

- Multiply each expansion by unknown coefficients (a, b, c, d, …) and add them
- Equate the coefficient of the desired derivative to 1 and coefficients of all other derivatives up to the desired accuracy to zero
- Write a matrix of coefficients and solve for unknowns
- Verify that the appropriate highest order term left has the desired power of (Δx)
- Study if error is dissipative or dispersive
- Symbolic manipulations can be used

Some High Order Stencils

$$\left.\frac{df}{dx}\right|_{i,j} = \frac{-\boldsymbol{f}_{\boldsymbol{i+2,j}} + \mathbf{8}\boldsymbol{f}_{\boldsymbol{i+1,j}} - \mathbf{8}\boldsymbol{f}_{\boldsymbol{i-1,j}} + \boldsymbol{f}_{\boldsymbol{i-2,j}}}{12\,\Delta x} + O(\Delta x^4)$$

$$\left.\frac{d^2f}{dx^2}\right|_{i,j} = \frac{-\boldsymbol{f}_{\boldsymbol{i+2,j}} + \mathbf{16}\boldsymbol{f}_{\boldsymbol{i+1,j}} - \mathbf{30}\boldsymbol{f}_{\boldsymbol{i,j}} + \mathbf{16}\boldsymbol{f}_{\boldsymbol{i-1,j}} - \boldsymbol{f}_{\boldsymbol{i-2,j}}}{12\Delta x^2} + O(\Delta x^4)$$

$$\left.\frac{d^3f}{dx^3}\right|_{i,j} = \frac{\boldsymbol{f}_{\boldsymbol{i+2,j}} - \mathbf{2}\boldsymbol{f}_{\boldsymbol{i+1,j}} + \mathbf{2}\boldsymbol{f}_{\boldsymbol{i-1,j}} - \boldsymbol{f}_{\boldsymbol{i-2,j}}}{\mathbf{2}\Delta x^3} + O(\Delta x^2)$$

$$\left.\frac{d^4f}{dx^4}\right|_{i,j} = \frac{\boldsymbol{f}_{\boldsymbol{i+2,j}} - \mathbf{4}\boldsymbol{f}_{\boldsymbol{i+1,j}} + \mathbf{6}\boldsymbol{f}_{\boldsymbol{i,j}} - \mathbf{4}\boldsymbol{f}_{\boldsymbol{i-1,j}} + \boldsymbol{f}_{\boldsymbol{i-2,j}}}{\Delta x^4} + O(\Delta x^2)$$

Compact Differencing

Higher order differencing stencils require more grid points. In complex flow geometries such stencils are not possible

Compact differencing schemes have limited support. The stencils are shorter and therefore easier near the boundaries. They are also higher order accurate. However, they require slightly more work.

Compact Differencing

Consider the following two stencils

$$\boldsymbol{f'_n = \left(\frac{D_0}{1 + \frac{1}{6}h^2 D_+ D_-}\right) f_n}$$

$$\boldsymbol{f''_n = \left(\frac{D_+ D_-}{1 + \frac{1}{12}h^2 D_+ D_-}\right) f_n}$$

$$\boldsymbol{D_0 f_n = \frac{1}{2h}(f_{n+1} - f_{n-1})}$$

$$\boldsymbol{D_+ f_n = \frac{1}{h}(f_{n+1} - f_n)}$$

$$\boldsymbol{D_- f_n = \frac{1}{h}(f_n - f_{n-1})}$$

Compact Differencing

To implement this, let us write

$$f'_n = F_n$$

$$f''_n = S_n$$

Then multiplying by the denominator

$$\frac{1}{6}F_{n+1} + \frac{2}{3}F_n + \frac{1}{6}F_{n-1} = \frac{1}{2h}(f_{n+1} - f_{n-1})$$

$$\frac{1}{12}S_{n+1} + \frac{5}{6}S_n + \frac{1}{12}S_{n-1} = \frac{1}{h^2}(f_{n+1} - 2f_n + f_{n-1})$$

We need to solve tridiagonal equations for F_n and S_n. Use these in parent equation like N-S equations to get f_n, the variable

Compact Differencing

$$F_n = f'_n - \frac{1}{180} h^4 f^{IV}$$

$$S_n = f''_n - \frac{1}{240} h^4 f^{VI}$$

If we use the regular finite difference stencil of higher order

$$F_n = D_0 \left(1 - \frac{1}{6} h^2 D_+ D_-\right) f_n$$

$$S_n = D_+ D_- (1 - \frac{1}{12} h^2 D_+ D_-) f_n$$

Compact Differencing

The truncation error here is

$$\boldsymbol{F_n = f'_n - \frac{1}{30} h^4 f^{IV}}$$
$$\boldsymbol{S_n = f''_n - \frac{1}{90} h^4 f^{VI}}$$

These expressions have same order of accuracy but larger stencils

Example of Compact Differencing (Hirsh, 1975)

Burgers Equation

$$u_t + \left(u - \frac{1}{2}\right)u_x = \nu u_{xx}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \left(u - \frac{1}{2}\right)F_i^m = \nu S_i^m$$

$$\frac{1}{6}\left(F_{i+1}^m + 4F_i^m + F_{i-1}^m\right) = \frac{1}{2\Delta x}\left(u_{i+1}^m - u_{i-1}^m\right)$$

$$\frac{1}{12}\left(S_{i+1}^m + 10S_i^m + S_{i-1}^m\right) = \frac{1}{\Delta x^2}\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right)$$

Example of Compact Differencing (Hirsh, 1975)

m can be n or (n+1) based on explicit or implicit

Explicit

$$\boldsymbol{C = \frac{\nu \Delta t}{2\Delta x} \leq (\frac{1}{6})^{1/2}}$$

$$\boldsymbol{\beta = \frac{\nu \Delta t}{\Delta x^2} \leq \frac{1}{2}}$$

Implicit

has no stability limit

Example of Compact Differencing

x	Exact	Kreiss(h=0.2)	F.D. (h = 0.2)	F.D. (h = 0.05)
		ν = 1/8		
0.0	0.50000	0.50000	0.5000	0.50000
0.2	0.68997	0.69033	0.6999	0.69054
0.4	0.83202	0.83224	0.8447	0.83276
0.6	0.91683	0.91687	0.9269	0.91744
0.8	0.96083	0.96082	0.9673	0.96123
1.0	0.98201	0.98199	0.9857	0.98225
1.2	0.99184	0.99182	0.9938	0.99197
1.4	0.99632	0.99631	0.9973	0.99638
1.6	0.99834	0.99834	0.9988	0.99838
1.8	0.99925	0.99925	0.9995	0.99927
		ν = 1/ 16		
0.0	0.50000	0.50000	0.5000	0.50000
0.2	0.83202	0.83825	0.9000	0.83224
0.4	0.96083	0.95933	0.9878	0.96082
0.6	0.99184	0.99216	0.9986	0.99182
0.8	0.99834	0.99809	0.9998	0.99834
1.0	0.99966	0.99974	1.0000	0.99966
1.2	0.99993	0.99989	1.0000	0.99993

Comparison of Computed Values of Solutions of Burgers' Equations with Exact Analytic Results

(Hirsh, 1975)

Driven Cavity (ψ-ω formulation)

Calculated Values of the Vorticity at the Moving (Upper) Wall for Various Methods

X =	0.0714	0.1428	0.2143	0.2857	0.3571	0.4286	0.5000	0.5714	0.6428	0.7143	0.7857	0.8571	0.9286
A	38.05	20.99	14.95	11.59	9.338	7.749	6.696	6.187	6.317	7.254	9.314	13.45	25.37
B	34.80	21.00	15.05	11.78	9.591	8.033	6.927	6.380	6.293	7.226	8.968	14.23	27.63
C	35.36	23.83	17.18	13.12	10.42	8.568	7.237	6.613	6.307	7.281	8.414	13.41	23.70
D	32.92	21.28	15.88	12.49	10.15	8.374	7.137	6.390	6.192	6.673	8.026	11.14	24.19
E	24.04	20.04	16.74	14.10	11.97	10.26	8.916	7.948	7.423	7.477	8.329	10.44	14.86

Note:

A – Second order ADI on 57×57 grid

B – Kreiss method with high order boundary conditions on 15×15 grid

C – Kreiss method with low order boundary conditions on 15×15 grid

D – Cubic spline method: R. S. Hirsh, SIAM 1974 Fall meeting, Oct. 23-25, 1974, Alexandria, VA.

E – Second order ADI on 15×15 grid

Higher Order Finite Volume Methods

Standard finite volume methods have following characteristics

We solve for average quantities over finite volumes

The flux balances give exact values for the averages

However, we <u>cannot</u> use the local values <u>as</u> averages in the cell flux calculations

Higher order Finite Volume Methods

Almgen, Aspden, Bell and Minion SIAM J. Sci. Comput (2013)

We need to reconstruct the cell face values from the cell average values and then compute the fluxes. This reconstruction must be done by assuming higher-order variations of the variables.

Notation

Define

$$\tilde{f}_{i+\frac{1}{2},j,k} = \frac{1}{h^2}\int_{face} f(x_{i+\frac{1}{2}}, y, z)\, dydz$$

$$\widetilde{\nabla}\phi_{i,j,k} = \{(\widetilde{\phi}_x)_{i+\frac{1}{2},j,k}, (\widetilde{\phi}_y)_{i,j+\frac{1}{2},k}, (\widetilde{\phi}_z)_{i,j,k+\frac{1}{2}}\}$$

Consider $Q_t + \nabla \cdot F(Q) = S$

Notation

$$h^3 \frac{d}{dt} Q_{i,j,k} +$$

$$\int_{i+\frac{1}{2},j,k} f_1\left(Q\left(x_{i+\frac{1}{2},j,k}, y, z\right)\right) dydz - \int_{i-\frac{1}{2},j,k} f_1\left(Q\left(x_{i-\frac{1}{2},j,k}, y, z\right)\right) dydz +$$

$$\int_{i,j+\frac{1}{2},k} f_2\left(Q\left(x, y_{i,j+\frac{1}{2},k}, z\right)\right) dxdz - \int_{i,j-\frac{1}{2},k} f_2\left(Q\left(x \quad , y_{i,j-\frac{1}{2},k}, z\right)\right) dxdz +$$

$$\int_{i,j,k+\frac{1}{2}} f_3\left(Q\left(x, y, z_{i,j,k+\frac{1}{2}}\right)\right) dxdy - \int_{i,j,k-\frac{1}{2}} f_3\left(Q\left(x, y, z_{i,j,k-\frac{1}{2}}\right)\right) dxdy +$$

$$\overline{S}_{i,j,k} h^3 \quad = \; 0$$

Cell face Average from Cell Center Averages

Given all averages, the cell face averages can be computed to fourth order accuracy as

$$\tilde{\phi}_{i+\frac{1}{2},j,k} = \frac{1}{12\Delta x}\left\{-\bar{\phi}_{i-1,j,k} + 7\left(\bar{\phi}_{i,j,k} + \bar{\phi}_{i+1,j,k}\right) - \bar{\phi}_{i+2,j,k}\right\}$$

Derived by integrating a one-dimensional interpolation formula.

Cell face Average from Cell Averages

$\left(\widetilde{\phi}_x\right)_{i+\frac{1}{2},j,k}$ **is required in the pressure Poisson equation**

$$\nabla \cdot \widetilde{\nabla}\overline{\phi}_{i,j,k} = \widetilde{\nabla} \cdot \widetilde{u}^*_{i,j,k}$$

Where $\overline{\phi}_{i,j,k}$ is pressure or potential; $\widetilde{u}^*_{i,j,k}$ is the $\overline{u}^*$ values computed at cell faces

$$\left(\widetilde{\phi}_x\right)_{i+\frac{1}{2},j,k} = \{\overline{\phi}_{i-1,j,k} + 15\left(-\phi_{i,j,k} + \phi_{i+1,j,k}\right) - \phi_{i+2,j,k}\}/(12\Delta x)$$

Cell face Average from Cell Averages

This gives a 13 point stencil for the operator or $\nabla \cdot \widetilde{\nabla}_h$

After solving the pressure-Poisson equation , the velocity is convected as

$$\widetilde{\boldsymbol{u}}_{i,j,k} = \widetilde{\boldsymbol{u}}^*_{i,j,k} - \widetilde{\nabla}^*_h \phi_{i,j,k}$$

And makes $\nabla \cdot \widetilde{\boldsymbol{u}}_{i,j,k}$ precisely zero

$$\overline{\boldsymbol{u}}_{i,j,k} = \overline{\boldsymbol{u}}^*_{i,j,k} - \overline{\nabla}_h \overline{\phi}_{i,j,k}$$

$$(\overline{\phi}_x)_{i,j,k} = \{\overline{\phi}_{i-2,j,k} + 8\left(-\overline{\phi}_{i-1,j,k} + \overline{\phi}_{i+1,j,k}\right) - \overline{\phi}_{i+2,j,k}\} /(12\Delta x)$$

Computing Nonlinear Terms

Primary difficulty is average of products is not equal to product of averages. We can include first order variations and get a fourth order scheme as

$$(\widetilde{\phi}\widetilde{\rho})_{i+\frac{1}{2},j,k} = \widetilde{\phi}_{i+\frac{1}{2},j,k}\widetilde{\rho}_{i+\frac{1}{2},j,k} + \frac{\Delta x^2}{12}\left(\widetilde{\phi}_x\widetilde{\rho}_y + \widetilde{\phi}_x\widetilde{\rho}_z\right) + O(h^4)$$

$$\widetilde{\phi}_y = \{-5\widetilde{\phi}_{i+\frac{1}{2},j+2,k} + 34\left(\widetilde{\phi}_{i+\frac{1}{2},j+1,k} - \widetilde{\phi}_{i+\frac{1}{2},j-1,k}\right) + 5\widetilde{\phi}_{i+\frac{1}{2},j-2,k}\}/(48\Delta x)$$

Diffusion Terms

$$\widetilde{\nabla} D(\boldsymbol{u}^*) = \boldsymbol{\nu} \widetilde{\nabla} \cdot \widetilde{\nabla} \boldsymbol{u}^*$$

$\widetilde{\nabla} \boldsymbol{u}^*$ is evaluated at cell faces using 4th order scheme. Then $\widetilde{\nabla} \cdot (\quad)$ is integrated over the control volume.

If an implicit method is used, it is brought to left hand side

Higher Order Temporal Schemes

The traditional fractional step is first order in time. It can be made second-order by using Adams Bashforth differencing scheme.

Range Kutta schemes provide higher order temporal accuracy by taking fractional time steps and combining solutions.

Higher Order Temporal Schemes

Consider the equation

$$\frac{du}{dt} = F(t, u)$$

$u(t, 0) = u^0$ **Then**

$$\Delta u' = \Delta t F(t^n, u^n)$$

$$\Delta u'' = \Delta t F(t^n + \frac{1}{2}\Delta t, u^n + \frac{1}{2}\Delta u')$$

$$\Delta u''' = \Delta t F\left(t^n + \frac{1}{2}\Delta t, u^n + \frac{1}{2}\Delta u''\right)$$

$$\Delta u'''' = \Delta t F(t^n + \Delta t, u^n + \Delta u''')$$

$$u^{n+1} = u^n + \frac{1}{6}(\Delta u' + 2\Delta u'' + 2\Delta u''' + \Delta u'''')$$

Higher Order Temporal Schemes

Stage 1

$$\widehat{\boldsymbol{u}}^{*} = \widehat{\boldsymbol{u}}^{\boldsymbol{n}} + \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{F}(\boldsymbol{t}^{\boldsymbol{n}}, \boldsymbol{u}^{\boldsymbol{n}}) - \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{G}(\boldsymbol{p}^{\boldsymbol{n}-(\boldsymbol{1}/\boldsymbol{2})})$$

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{u}}^{*} = \boldsymbol{0}$$

$$\widehat{\boldsymbol{u}}' = \widehat{\boldsymbol{u}}^{*} - \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{G}(\boldsymbol{p}' - \boldsymbol{p}^{\boldsymbol{n}-(\boldsymbol{1}/\boldsymbol{2})})$$

Stage 2

$$\widehat{\boldsymbol{u}}^{**} = \widehat{\boldsymbol{u}}^{\boldsymbol{n}} + \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{F}\left(\boldsymbol{t}^{\boldsymbol{n}+\boldsymbol{1}/\boldsymbol{2}}, \boldsymbol{u}'\right) - \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{G}(\boldsymbol{p}')$$

$$\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{u}}^{**} = \boldsymbol{0}$$

$$\widehat{\boldsymbol{u}}'' = \widehat{\boldsymbol{u}}^{**} - \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}}\boldsymbol{G}(\boldsymbol{p}'' - \boldsymbol{p}')$$

Higher Order Temporal Schemes

Stage 3

$$\widehat{\boldsymbol{u}}^{***} = \widehat{\boldsymbol{u}}^{n} + \Delta t \boldsymbol{F}\left(t^{n+1/2}, \boldsymbol{u}''\right) - \Delta t \boldsymbol{G}(\boldsymbol{p}'')$$

$$\nabla \cdot \widehat{\boldsymbol{u}}^{***} = 0$$

$$\widehat{\boldsymbol{u}}''' = \widehat{\boldsymbol{u}}^{***} - \Delta t \boldsymbol{G}(\boldsymbol{p}''' - \boldsymbol{p}'')$$

Stage 4

$$\widehat{\boldsymbol{u}}^{****} = \widehat{\boldsymbol{u}}^{n} + \frac{\Delta t}{6}\boldsymbol{F}(t^{n}, \boldsymbol{u}^{n}) + \frac{\Delta t}{3}\boldsymbol{F}\left(t^{n+1/2}, \boldsymbol{u}'\right) + \frac{\Delta t}{3}\boldsymbol{F}\left(t^{n+1/2}, \boldsymbol{u}''\right) + \frac{\Delta t}{6}\boldsymbol{F}\left(t^{n+1}, \boldsymbol{u}'''\right) - \Delta t \boldsymbol{G}(\boldsymbol{p}''')$$

$$\nabla \cdot \widehat{\boldsymbol{u}}^{****} = 0$$

$$\widehat{\boldsymbol{u}}^{n+1} = \widehat{\boldsymbol{u}}^{****} - \Delta t \boldsymbol{G}(\boldsymbol{p}^{n+1/2} - \boldsymbol{p}''')$$